PHYS 302 Spring 2014 3rd Homework Assignment. Due Wed March 12 at Lecture.

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1. In this walk through we shall see the structure of rotations in more detail. In fact we shall be studying the so called rotation group. I hope at the end you will have a better view of the relation between rotations and cross product in 3-dimensional space. Remember that in general matrix multiplication is not commutative: That is in genetal $AB \neq BA$ if they are matrices.

Consider a cartesian co-ordinate system with right handed ortho-normal unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. A right handed rotation about the $\hat{k}$ axis of a point with co-ordinates $\{x, y, z\}$ can be written as a matrix relation. The resulting co-ordinates with respect to the original set of axis are given to be

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where for brevity we have $c = \cos(\theta)$ and $s = \sin(\theta)$. Call the matrix above $\mathcal{R}_z(\theta)$ and show that it satisfies $\mathcal{R}_z^T \mathcal{R}_z = \mathcal{R}_z \mathcal{R}_z^T = \mathrm{Id}$, where Id is the 3x3 unit matrix and T refers to transposition -these matrices are called special orthogonal matrices in 3 dimensions. Show also that $\det(\mathcal{R}_z) = 1$.

Now calculate J_z defined as

$$J_z \equiv -\frac{1}{i} \frac{d\mathcal{R}_z}{d\theta}|_{\theta=0}$$

Is the matrix J_z hermitean? That is does taking its transpose and taking complex conjugation of its elements reproduce the same matrix? Now show EXPLICITLY that one can have

$$\mathcal{R}_z(\theta) = \exp\left(-i\theta J_z\right)$$

Now define the following matrices

$$\mathcal{R}_x(\theta) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

and

$$\mathcal{R}_y(\theta) \equiv \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}$$

What can you say about the apparent difference of the sign convention in $\mathcal{R}_y$. The answer is somewhere in the text above. Check your reasonning.

Thus just like before we can define the following matrices

$$J_x \equiv -\frac{1}{i} \frac{d\mathcal{R}_x}{d\theta}|_{\theta=0}$$

$$J_y \equiv -\frac{1}{i} \frac{d\mathcal{R}_y}{d\theta}|_{\theta=0}$$

and eventually have $\mathcal{R}_x(\theta) = \exp\left(-i\theta J_x\right)$ and $\mathcal{R}_y(\theta) = \exp\left(-i\theta J_y\right)$.

Now we come to the crux of the matter. Define the following object -where for simplicity every rotation is made via the same argument-

$$\mathcal{M}(\theta) \equiv \mathcal{R}_x(-\theta)\mathcal{R}_y(-\theta)\mathcal{R}_x(\theta)\mathcal{R}_y(\theta)$$

Show that $\mathcal{M}(\theta)$ is also a special orthogonal matrix in three dimensions.

Now assume θ is small and use the expressions for $\mathcal{R}_x$ and $\mathcal{R}_y$ in terms of J_x and J_y, and expand both sides of the equation to first non-trivial order in θ. That is since the LHS has the following form

$$\mathcal{M}(\theta) = \mathrm{Id} + \theta\mathcal{A} + \frac{1}{2}\theta^2\mathcal{B}\cdots$$

All you have to do is to expand the RHS and match the like powers of θ. You should get $\mathcal{A} = 0$ can you argue about this without making a computation but using the orthogonality of M?.

So there remains to identify B, do it. Hint: it must contain $J_xJ_y - J_yJ_x$.

Thus the commutators of these J matrices which are rightfully called generators -because given a parameter they generate rotation matrices upon exponentiation, are important objects. To that end calculate

$$[J_x, J_y]$$
$$[J_z, J_x]$$
$$[J_y, J_z]$$

and show that all of these commutation relations can be written as

$$[J_a, J_b] = i\epsilon_{abc}J_c$$

where the indices run from 1 to 3 and we have used the summation convention. It is amazing that the epsilon tensor appears here and provides a direct meaning to the existence of CROSS PRODUCT of two vectors. You should also make a connection here to the previous homework assignment: There you showed that an orthogonal matrix can be written as $\exp(A)$ where A is anti-symmetric, but an anti-symmetric matrix in three dimensions require only three numbers and thus can be written as $A_{ab} = \epsilon_{abc}\beta_c$.

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2. Remember that the Euler-equations for force and torque free motion (total angular momentum is conserved) of a rigid body about its center of mass can be expressed in body-rotating co-ordinates as follows

$$\begin{aligned}
I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\
I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\
I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2
\end{aligned}$$

Again, these components refer to the co-ordinate system which rotates with the body and the axes are aligned with the principal axes of the body. That is those axes are eigenvectors of the moment of inertia matrix calculated in reference to the center of mass.

If $I_1 = I_2 = I_3$ the solution is simple: $\vec{\omega}$ is a constant vector, and since the moment of inertia is a multiple of identity $\vec{\omega}$ and $\vec{L}$ are collinear. So the constancy of $\vec{\omega}$ is related to the fact that $\vec{L}$ is a conserved vector.

2-1. Now let us make it a tad more complicated. Assume that $I_1 = I_2 \equiv I$ and $I_3 \equiv \tilde{I} \neq I$. The equations take the following form

$$\begin{aligned} I\dot{\omega}_1 &= (I - \tilde{I})\omega_2\omega_3 \\ I\dot{\omega}_2 &= (\tilde{I} - I)\omega_3\omega_1 \\ \tilde{I}\dot{\omega}_3 &= (I - I)\omega_1\omega_2 = 0 \end{aligned}$$

Since ω_3 is a constant these equations are no longer non-linear and an exact solution can be found. Define $\omega_3 \equiv w$ and assume the initial conditions $\omega_1(0) = a$ and $\omega_2(0) = b$. Also define the quantity $w(I - \tilde{I})/I \equiv \Omega$. Find $\omega_1(t)$ and $\omega_2(t)$, conforming to the given initial conditions. Hint: There are various approaches for this such as defining a complex quantity $z \equiv \omega_1 + i\omega_2$ or writing the equation for ω_1 and ω_2 by using a 2x2 matrix or taking further derivatives to finally arrive at separate equations for ω_1 and ω_2. Use one that you like. But for the last approach you will have increased the degree of the equation, so remember that it is the initial first order equation that is to be solved.
Does the signature of Ω or the relative strengths of I and $\tilde{I}$ matter?

2-2. There remains the most general case. Let us pick an ordering $I_3 > I_2 > I_1$ and keep this fixed for this problem. The equations are now genuinely non-linear and an exact solution -in the sense of exact analytic expressions- is not present. However we still can extract valuable informations.

Now let us assume the following ansatz $\omega_3 = w$ which is a constant while $\omega_1(t) = a(t)$ and $\omega_2(t) = b(t)$. We shall be looking for cases where $a(t) << w$ and $b(t) << w$, that is look for conditions for which this will be valid. To that end ignore all quadratic expressions such as a^2, b^2 and ab. Show that the equations approximately take the form

$$\begin{aligned} I_1\dot{a} &= (I_2 - I_3)w\ b \\ I_2\dot{b} &= (I_3 - I_1)w\ a \\ I_3\dot{\omega}_3 &\approx 0 \longrightarrow \omega_3 \equiv w \end{aligned}$$

Now that we fixed the strengths of the eigenvalues of the moment of inertia tensor the question is : Do a and b remain small given that at $t = 0$ they were small compared to w? Answer this.

Now repeat the procedure for all the other axes and aswer the same question. That is first let $\omega_2 = w$ a constant, $\omega_3 = c(t)$ and $\omega_1(t) = a(t)$. If initially c and a are much smaller than w do they remain small? Repeat for $\omega_1 = w$, $\omega_2 = b(t)$ and $\omega_3 = c(t)$ in a similar fashion.

Now combine all your results and argue on the stability -that is perturbations remain small- of single axis rotations of a rigid body.

2-3. A general notion about torque free rotations. Remember that the rotational kinetic energy of a rigid body can be written as $K_{\text{rot}} = \frac{1}{2}\omega_a I_{ab}\omega_b$. But also in the form $K_{\text{rot}} = \frac{1}{2}\vec{\omega} \cdot \vec{L}$. This quantity is a constant of motion. From the perspective of an inertial frame $\vec{L}$ is a constant vector. What does this say about $\vec{\omega}$ in the inertial frame? Your answer must be clear and general. Repeat the same analysis from the perspective of the body-rotating frame remember that there are differences in the time derivative of a vector between the frames.

2-4. Now the kinetic energy of the rigid body can be written as

$$\frac{1}{2}I_1\omega_1(t)^2 + \frac{1}{2}I_2\omega_2(t)^2 + \frac{1}{2}I_3\omega_3(t)^2$$

Since this is a constant this means that the values of $\omega_1(t)$, $\omega_2(t)$ and $\omega_3(t)$ travel on the surface of an ellipsoid, which is called the Poinsot's ellipsoid. As a side dish, do some network research and define the meanings of prolate and oblate, and classify a coin and a cigar as such.

Now this means that the values of ω_a can not increase indefinitely if the kinetic energy is to be conserved. Describe the geometrical meaning of the results you have found in 2-1 and 2-2 in terms of this language.

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3. Read the entire chapter of Kleppner and Kolenkow on rigid body motion and solve the problems 7.3 and 7.4. (These were assigned last semester but only as the last homework so they will serve as a nice review.)