PHYS 401 Fall 2019 1'st homework assignment. Due Oct 7'th.

In what follows always use the index notation we have introduced during the flectures. Here is a quick review. An orthonormal basis obeys the following

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

where δ_{ij}, called the Kronecker symbol vanishes whenever $i \neq j$ and it is normalized to unity whenever $i = j$. The indices take values from the set $\{1, 2, 3\}$ and refer to a choice of orthogonal directions at a given point in space.

An orthonormal (and right handed! -understand why-) basis set obeys the following

$$\begin{aligned} \hat{e}_1 \times \hat{e}_2 &= \hat{e}_3 \\ \hat{e}_3 \times \hat{e}_1 &= \hat{e}_2 \\ \hat{e}_2 \times \hat{e}_3 &= \hat{e}_1 \end{aligned}$$

We have also defined the Levi-Civita symbol ϵ_{ijk} which is completely antisymmetric (whenever one interchanges two indices one picks a minus sign) and normalized to unity such that one has $\epsilon_{123} \equiv +1$.

We have argued that $\epsilon_{ijk} = (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k$. Verify it explicitly.

Within the index notation a vector is expanded over the basis as a sum

$$\vec{A} = \sum_{i=1}^{3} A_i \hat{e}_i$$

where

$$A_i \equiv \vec{A} \cdot \hat{e}_i$$

However as the number of vectors increase in an expression the summation signs grow in number in an annoying manner. To deal with this one can use Einstein's summation convention. In this convention a repeated index is understood to be summed over from 1 to 3 and the summation sign is omitted. For instance one can write

$$\vec{A} = A_i \hat{e}_i.$$

On the other hand sometimes one may refer to instances where the summation is not intended. In those cases one explicitly mentions that there is *no sum on* whatever indices are not summed.

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1. Show that $\vec{A} \cdot \vec{B} = A_i B_i$.
2. Show that $(\vec{A} \times \vec{B}) \cdot \hat{e}_k = \epsilon_{ijk} A_i B_j$.
3. Prove the identity
$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$
4. Show that the magnitude of $\vec{A} \cdot (\vec{B} \times \vec{C})$ is the volume of the paralleliped defined by the given vectors.

5. Prove the identity
$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

6. Define $|\vec{A}| \equiv \sqrt{\vec{A} \cdot \vec{A}}$ and show the following inequalities,
$$\begin{aligned} \vec{A} \cdot \vec{B} &\leq |\vec{A}|\,|\vec{B}| \\ |\vec{A} + \vec{B}| &\leq |\vec{A}| + |\vec{B}| \end{aligned}$$
The first one is called the Cauchy-Schwarz inequality and the second is called the triangle inequality. You may use the definition $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos(\theta)$ where θ is the angle between the vectors when the tips are joined. Do you need it?

7. Show that
$$(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = |\vec{A}|^2\,|\vec{B}|^2 - (\vec{A} \cdot \vec{B})^2.$$

8. Visit

`https://en.wikipedia.org/wiki/Vector_algebra_relations`

and using the notation there prove the last item just before references; the one about representing an arbitrary fourth vector in terms of given three non-collinear vectors.

9. Remember the gradient operator in cartesian co-ordinates we have introduced in lecture
$$\vec{\nabla} \equiv \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$
Show that if this hits a function $V(r)$ where $r \equiv \sqrt{x^2 + y^2 + z^2}$, the resulting vector field is always central; along the radial lines.

10. Consider the question above where V can now be any function of the positions; $V(x, y, z)$. Consider the curves in space defined as $V(x, y, z) = V_o$ where $V_o \in \mathcal{R}$ is any real number. Show that the vector field $\vec{\nabla}V$ is perpendicular to those curves at any given point.