PHYS 401 Fall 2019 6'th homework assignment. Due Nov 25'th.

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1. Solve problems 5.1, 5.5, 5.6 and 5.7.
2. Solve problems 5.8,5.9, 5.10.
3. Solve problem 5.13.
4. Solve problem 5.14,5.15,5.16 and 5.17.

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5. NOT MANDATORY SOLVE IT FOR EXTRA CREDITS.

   In this directed exercise we shall consider the cases where due to Helmholtz theorem the use of

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_o} \int d^3\vec{r'} \, \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} \tag{1}$$

   is not justified, whereas the use of

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_o} \int d^3\vec{r'} \, \frac{\rho(\vec{r'})(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3} \tag{2}$$

   is still feasible. This may seem contradictory but note that the use of $\vec{E} = -\vec{\nabla} V$ requires that the first integral converges whereas the direct use of the second integral requires a different type of convergence.

   To explore this let us consider a line of charge sitting along the z axis such that one has

$$\rho(\vec{r}) = \lambda\delta(x)\delta(y)$$

   Now of course using the integral form of Gauss's law one gets the result rather quickly. Show that one gets

$$\vec{E} = \frac{\lambda\hat{\rho}}{2\pi\rho}$$

   where $\rho = \sqrt{x^2 + y^2}$ and $\vec{\rho} = x\hat{\imath} + y\hat{\jmath}$, but remember that to apply the integral form of the Gauss' law one needs a cylinder and one has the cancellation of the length of the cylinder from both sides!

   Now using the ρ given find the electric field directly from the expression in Eq.2 and show that it yields the correct result.

Now try using the same ρ with the expression of Eq.1 and show that the integral does not converge; it is ill defined. This will eventually come to the fact that $\int_{-\infty}^{\infty} du/\sqrt{1+u^2}$ blows up, albeit only logarithmically.

To control this divergence in the potential and to be in accord with Helmholtz theorem we need a charge density that vanishes at infinities. To that end introduce the following

$$\rho(\vec{r}) = \lambda\delta(x)\delta(y)\Theta(L-z)\Theta(z+L)$$

which simply means that, now the line of charge extends from $z = -L$ to $z = L$.

Now use this given charge density along with Eq.1, and show that it will be given as

$$V(\vec{r}) = \frac{\lambda}{4\pi\epsilon_o}\left[\sinh^{-1}\left(\frac{L-z}{\rho}\right) + \sinh^{-1}\left(\frac{L+z}{\rho}\right)\right] \tag{3}$$

This issue is of course the following: Taking $L \to \infty$ and then applying the gradient is one thing, and applying the gradient and taking $L \to \infty$ is another. To that end find $\vec{E} = -\vec{\nabla}V$ using the potential in Eq.3. Now consider the limit $L \to \infty$ after applying the gradient and show that one again gets $\vec{E} = \lambda\hat{\rho}/2\pi\epsilon_o\rho$.

But you may say is there no potential for the infinite line of charge? Certainly there is one since one can read the relation between the electric field and the potential from the line integral as well,

$$V_b - V_a = -\int_a^b d\vec{r'} \cdot \vec{E}(\vec{r'})$$

Show that this definition will yield

$$V(\vec{r}) = V(\vec{r_o}) + \frac{\lambda}{2\pi\epsilon_o}\log(\frac{\rho_o}{\rho})$$

Now to connect this with the result you found for the finite line first remember that

$$\sinh^{-1}(u) = \log(x + \sqrt{1+x^2})$$

and then consider first the exact expression for $V(\vec{r}) - V(\vec{r_o})$ and only then consider the limit $L \to \infty$ and show that one gets logarithmic dependence of the potential we have just found.

But, how, you may say, that the infinite line of charge has some sort of a solution by any means -the integral form of the Gauss' law in this case. Since the construct is clearly unphysical. There are two nice answers to this. First consider the exact result you have found for the finite line of charge and assume that both ρ and z are much less than the lenght of the wire, which is $2L$. This means that we are close to the wire and perceive it as an infinite one. Show that a limit calculated as this gives the result for the infinite line.

The second answer relies on the very high symmetry of the situation. For an infinite straight line of charge the problem seems effectively two dimensional and thus one is tempted to solve the Poisson's equation in two dimensions. Since the system does not have angular dependence we need

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}$$

Now you have already seen that the list of solutions to $\nabla^2 V = 0$ in 2-dimensions are given as a constant function or $\log(\rho)$. That is a general solution would be given as $V = V_o + \log(\rho_o/\rho)$. Of course this result is not valid when $\rho = 0$ and using the generalization of the Gauss' law two two dimensions we will also get the correct prefactor $\lambda/2\pi\epsilon_o$. Here the idea is that in the cross section problem the charge (its 2D equivalent) sits at the center and the charge density (its 2D equivalent) vanishes at the infinities -of the given plane.

When the wire is not straight one can not have the equivalence of all planes that cut the wire orthogonally and the answer can only be found using Eq.2. After which one may find the potential using the line integral of the electric field.