PHYS 58F HWA1. Due Mon Feb 24'th at lecture hours.

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2. Solve problems 2.2.1,2,3,4,5,6; 2.3.4,8,9.

4. In this problem we assume we are in a linear vector space over the complex numbers as in quantum mechanics. We know that the states of the system live in this space and they are called the kets, $|\psi\rangle$. If we have an inner product in this space -which is essentially a metric- defined as follows

$$\begin{aligned}
(|\psi\rangle, |\phi\rangle) &= (|\psi\rangle, |\phi\rangle)^* && \text{(1a)}\\
(|\psi\rangle, |\phi_1\rangle + \lambda|\phi_2\rangle) &= (|\psi\rangle, |\phi_1\rangle) + \lambda\,(|\psi\rangle, |\phi_2\rangle) && \text{(1b)}\\
(|\psi_1\rangle + \lambda|\psi_2\rangle, |\phi\rangle) &= (|\psi_1\rangle, |\phi\rangle) + \lambda^*\,(|\psi_2\rangle, |\phi\rangle) && \text{(1c)}\\
(|\psi\rangle, |\psi\rangle) &\geq 0. \quad 0 \text{ iff } \quad |\psi\rangle = 0 && \text{(1d)}
\end{aligned}$$

Now consider the dual space to the ket space, which are defined as the linear space of functionals over the kets. That is the elements of the dual space, say χ are maps from the kets to the reals: $\chi(|\psi\rangle) \equiv \langle\chi|\psi\rangle \in \mathbb{C}$. Let us call those elements as bras.

Now using the inner product one can associate a bra for any given ket as we have seen in class for linear spaces. Here we have the extra situation that the vector spaces are over the complex field of numbers. The association is provided with

$$\langle\phi|\psi\rangle = (|\phi\rangle, |\psi\rangle) \qquad \text{provides} \quad |\phi\rangle \longrightarrow \langle\phi| \tag{2}$$

Note that the association is anti-linear in the sense that the correspondence obeys

$$\lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle \longrightarrow \lambda_1^*\langle\phi_1| + \lambda_2^*\langle\phi_2| \tag{3}$$

The converse question is the following: *Is there a ket for any given bra?*

Let us consider for simplicity the space of square integrable physical states in one physical dimensions. We know that the well known inner product in this case is given as

$$(|\psi_1\rangle, |\psi_2\rangle) = \int_{-\infty}^{\infty} dx\ \psi_1(x)^*\psi_2(x)$$

where we require that for each element in this space one must have

$$(|\psi\rangle, |\psi\rangle) = \int_{-\infty}^{\infty} dx\ |\psi(x)|^2 < \infty$$

with this definition positivity is guaranteed.

As usual define the bras associated with kets using (2)[1].

Now let us define the following family of functions

$$f_\epsilon(x) = \begin{cases} 1/\epsilon & \text{for} \quad |x| \leq \epsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

Argue that there is a ket $|f_\epsilon\rangle$ associated with this function which is square integrable for $\epsilon \neq 0$. Similarly show that there thus corresponds a bra for this ket.

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[1] But recall that bras actually are functionals over the kets

Now consider the case $\epsilon = 0$. Show that one can associate with it a bra (that is a functional over the kets) but there is no ket associated with that bra. Can you think of physical examples very directly related to the example here?
5.*Abstract index notation.* Read and understand pages 239-243 from the book by Roger Penrose called The Road to Reality.