PHYS 58F HWA2. Due Mon Mar 16'th at lecture hours.

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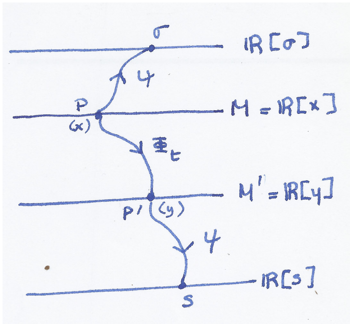
1. Solve problems 3.1.7,8,9.
2. Read section 3.2 (start by reading 3.1.11) on induced metrics and solve problems 3.2.2,3,4,8.
3. In addition to the above consider an embedding of S^1 into $\mathbb{R}^2$ and find the induced metric on S^1 due to the standard metric on the Euclidean plane.
4. Consider $\mathbb{R}^2$ with the special relativistic metric η (the Minkowski space $\mathbb{M}$) given as

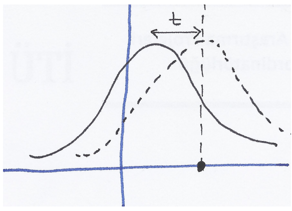
$$\eta = \eta_{ij} dx^i \otimes dx^j = dx^1 \otimes dx^1 - dx^2 \otimes dx^2$$

Now perform a map from S^1 to this space, as if you would do to ordinary Euclidean plane, and find the induced metric on S^1. In some regions for the co-ordinate on S^1 which we shall call ϕ the metric will be positive and in some regions it will be negative. Try to make a connection to the space-like, time-like and null intervals in special relativity.
5. Now consider the map from $T^2 \approx S^1 \times S^1$ to $\mathbb{M} \times \mathbb{M}$ generalizing the above example. Find the metric induced on the Torus. Note that in $\mathbb{M} \times \mathbb{M}$ there will be 2 minus signs in the metric. Compare this to one of the previous questions.
6. Now consider a light source sitting at the origin at $\mathbb{R}^2$, for our purposes these are rays coming out of origin. Assume one is to walk in such a way that the direction of the walk always has constant angle α in relation to the outcoming rays. Show that this results in a spiral motion away or toward the origin. Try to use the notation we are using thus far in defining vectors and their integral curves. After that make a point on why moths would tend to fly near an artificial light source at night. Do you think they are in "*love*" with the light now?
7. Consider $M = \mathbb{R}[x]$ and a vector field on it given as $V = \partial_x$. Show that the integral curves of this vector field is in the co-ordinate basis as $x(t) = x_o + t$, hence the vector field is complete. Now consider the vector flow map of this vector field, $\Phi_t : M \to M$. For convenience you can call the image manifold $M' = \mathbb{R}[y]$. Show that the action of the flow is given in the co-ordinate basis as $y = x + t$ -recall that the flow map is along the direction of the integral curve. Now imagine a function $\psi : M' \to \mathbb{R}[s]$, as we know the pull-back map on M is given as $\Phi_t^* \psi = \psi \circ \Phi_t$. Show that one gets $s = \psi(x + t)$ under this map. Now consider a copy of the function ψ this time from $M \to \mathbb{R}[\sigma]$ we now that in the co-ordinate representation it is given as $\sigma = f(x)$. We can thus see the situation as if we have two different functions defined at $P \in M$ -with co-ordinate x- and we would like to compare them. This is one way of seeing the effect of the Lie derivative: Show of course that in this case we have $\mathcal{L}_V \psi = V\psi$ which in the co-ordinate basis gives $\partial_x \psi(x) = \partial\psi(x)/\partial x$. In the following there is a diagram of this situation,



Digression: To be strict we do not need to use pull-backs to compare functions at different points of a manifold. True, we have identified $\mathcal{T}^0_0(M) = \mathcal{F}(M)$ but this was mainly to have a combined prescription, with general tensor fields. The recipe is simple, say we would like to compare the values of a function ψ at different points on the manifold P and Q all we have to do is to subtract from ψ a function which is a multiple of identity, that is $f(Q)\mathrm{Id}_M$. This way we get a new function which gives zero at the point Q and the other values are just the differences. Clearly we can not do this for instance for vector fields because not every manifold would allow for a constant vector field (remember the fact that there can be no vector field without a zero on the sphere). Nevertheless it is a good practice to use pull-backs for functions.

7.1 Yet another way to visualize the pull-back of functions on manifolds is via the graph of the function. For simplicity we shall consider the case where $M = \mathbb{R}[x]$. Now consider a function $\psi : M \to \mathbb{R}[\sigma]$. The graph of a function can be defined as the map from $M \times \mathbb{R}[\sigma] \to \mathbb{R}^2[x_1, x_2]$ where one simply posits $(x, \sigma) \mapsto (x_1, x_2)$. Now let us again recall the vector field above and its flow to a $M' = \mathbb{R}[y]$ and the associated pull-back of the copy of ψ this time from $M' \to \mathbb{R}[s]$. Similarly one can also construct the graph of this pulled back object from $M' \times \mathbb{R}[s] \to \mathbb{R}^2[x_1, x_2]$. One gets the following graph



Note that the solid curve is the graph of the pulled back function, and it is slided against the flow of the vector field ∂_x. You should also read 4.2.2, which is along the same lines, I have tired to make the exposition a bit redundant to clarify it.

8. Now replace $V = x\partial_x$ in problem 1. Find first the integral curves of V. Than assume a function ψ on M and find its pull-back and represent it in the co-ordinate basis and finally compute $\mathcal{L}_V\psi$. Incidentally consider the finite difference $\Phi_t^*\psi - \psi$ and represent it in the co-ordinate basis. This is related to what is called the q-derivative -you can further elaborate with M.Arık, if you are interested- of a function which is given as

$$D_q f = \frac{f(qx) - f(x)}{q - 1}$$

which in the limit $q \to 1$ becomes the ordinary derivative times x. Of course in the previous problem the associated object is the discrete difference given as $(f(x+t) - f(x))/t$. So you can have you beloved definition of a derivative but a vector field will be associated to it.

9. Now consider the vector fields we have so far seen, $V = \partial_x$ and $W = x\partial_x$. Consider the flows defined for each and calculate the effect of the following composition of flow on a point on the manifold M with co-ordinate x.

$$\Phi_{V\,t} \circ \Phi_{W\,t} \circ \Phi_{V\,-t} \circ \Phi_{W\,-t}$$

Now set $t = \epsilon$ and expand to the first non-trivial order in ϵ.

Now calculate $U = \mathcal{L}_V W$ -which is of course $-\mathcal{L}_W V$- which is a vector field itself. Study the effect of the flow $\Phi_{U\,\tau}$ on a point on the manifold M with co-ordinate x and make a connection to τ and ϵ to first non-trivial order.

10. Read 4.1.10, 4.1.11, 4.1.12 and solve 4.1.14.

For what follows also read 4.2.3 for the visualization of the pull back of vector fields under diffeo-morphisms.

11. Consider $M = \mathbb{R}^2[x, y]$ and the vector fields $V = \partial_x$ and $W = x\partial_x + y\partial_y$ on them. First find the integral curves of V and W. Now pull back W in relation to the flow generated by V, and calculate the finite difference ${\Phi_{V\,t}}^* W - W$. Geometrically represent this difference vector, using ordinary $\hat{\imath}$ and $\hat{\jmath}$ unit vectors we use on the plane. Now consider setting $t = \epsilon$ and express the lowest non-trivial contribution for the above. Now calculate $\mathcal{L}_V W = [V, W]$ and compare the results.

12. Repeat the above with $W = x\partial_y - y\partial_x$ and $V = \partial_x$.

13. Now consider $M = S^2$, the two dimensional sphere. Construct vector fields associated with the meridians and parallels; call them V and W respectively. Note that meridians do not intersect among themselves (similarly for the parallels) and hence they tear up the manifold. Use the embedding in 3.2.4. After this calculate $\mathcal{L}_V W$ and interpret in terms of the parallelograms language.

14. Remember the description of the torus $T^2 = S^1 \times S^1$ in 1.5.11. Using the boundary conditions on the square consider the family of straight lines starting from a point, with a definite slope. Which ones close after a finite parameter travel along the curves? Are there curves that never close on themselves? If so can you say they constitute a single vector field that tear up the torus? Remember the relative densities of subsets of real numbers: integers, fractions, irrationals and trancendentals.