Intersection democracy for winding branes and stabilization of extra dimensions

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Received 12 February 2005; received in revised form 31 May 2005; accepted 19 June 2005
Available online 28 June 2005

Abstract

We show that, in the context of pure Einstein gravity, a democratic principle for intersection possibilities of branes winding around extra dimensions in a given partitioning yield stabilization, while what the observed space follows is matter-like dust evolution. Here democracy is used in the sense that, in a given decimation of extra dimensions, all possible wrappings and hence all possible intersections are allowed. Generally, the necessary and sufficient condition for this is that the dimensionality \( m \) of the observed space dimensions obey \( 3 \leq m \leq N \) for \( N \geq 3 \), where \( N \) is the decimation order of the extra dimensions.

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1. Introduction

One extension of standard theories of high energy physics possibly unifying all known forces is string theory which mathematically necessitates extra dimensions other than the three we observe daily and that these extra dimensions are compact and very small. One important question once this proposition is accepted is that why these extra dimensions remained so small in contrast to the known universe. It is therefore important to look for ways to stabilize the size of extra dimensions within the context of cosmological evolution of the universe where it is known that the observed dimensions expand and have expanded throughout its history.

The literature on this nowadays rather vivid topic is considerable and we refer the reader to the following articles on brane gas cosmology [1–23]. Articles [1–7] particularly deal with the stabilization problem, while [8–23] are works on brane gas cosmology also relevant to this work.

In this Letter we are taking the full manifold of extra dimensions \( \mathcal{M} \) to be a product manifold \( \mathcal{M} = \prod_i^N \mathcal{M}_i \) and study the stabilization of the overall sizes of each \( \mathcal{M}_i \) and consequently that of \( \mathcal{M} \).\(^1\) Thus we do

\(^1\) In this work, from now on “stabilization of extra dimensions” should be considered as the stabilization of the overall sizes of the partitionings \( \mathcal{M}_i \) of the full manifold of extra dimensions \( \mathcal{M} \).
not consider the stabilization of the internal structure of each $M_i$. For simplicity we will confine the study to pure Einstein gravity only and consequently we assume that in the string theory framework the dilaton is stabilized (with some similar mechanism or otherwise). On the other hand we consider only brane winding modes for studying stabilization. The main purpose of this Letter is to introduce the idea of democratic winding which is basically suggesting that one should consider all the possible windings of branes around the partitionings of the extra dimensions and use it to study the consequences of the requirement for stabilization on the dimensionality of the observed space.

The outline of the manuscript is as follows. After laying out the main mathematical formalism we present two explicit cases of winding schemes. Namely the 2-fold and 3-fold decimations, where the extra dimensions are divided into two and three lumps, respectively. After discussing the possibilities for stabilization of extra dimensions in these explicit examples we then present stabilization conditions for an $N$-fold decimation where we made the simplifying assumption of symmetric decimation, that is the extra dimensions are divided into $N$ Ricci flat parts of same topology and same dimensionality $p$. Thus the dimensionality of the full manifold of extra dimensions is $Np$. This assumption makes it easy to deal with an otherwise very complicated non-linearly coupled system of equations. The general result is that for stabilization to occur the number of dimensions of the observed space has to obey $3 \leq m \leq N$ (for $N \geq 3$). This result does not depend on the dimensionality of the partitionings $p$ and hence hints at a stabilization solution for a non-symmetric decimation of the full manifold of extra dimensions. Furthermore, this result being independent of $p$ also covers the case $p = 1$ which would mean in our context that $M_E \equiv (S_1)^N$ (that is each partitioning is a circle) and consequently constitutes an example of the stabilization of the full shape moduli.

2. Formalism

The metric relevant for cosmological purposes is given by the following,

$$ds^2 = -dt^2 + e^{2B(t)} dx^2 + \sum_i e^{2C_i(t)} dy_i^2. \quad (1)$$

Here the $C_i$ and $y_i$ represent the scale factors and the coordinates of extra dimensions, respectively. The dimensionality of each partition is $p_i$. For clarity we separated the observed dimensions with scale factor $B$ and dimensionality $m$. The observed dimensions are taken to be flat following the observational fact that the universe is flat. The total space–time dimensionality is $d = 1 + m + \sum_i p_i$.

The pure Einstein gravity equations coupled to matter is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 T_{\mu\nu}. \quad (2)$$

With these assumptions the equations of motion for the scale factors can be cast as follows (we set $\kappa^2 = 1$)

$$\ddot{A} = m \dot{B}^2 + \sum_i p_i \dot{C}_i^2 + 2\rho, \quad (3a)$$

$$\ddot{B} + \dot{A} \dot{B} = T_{bb} - \frac{1}{d-2} T, \quad (3b)$$

$$\ddot{C}_i + \dot{A} \dot{C}_i = T_{ci} - \frac{1}{d-2} T, \quad (3c)$$

$$A = m B + \sum_i p_i C_i. \quad (3d)$$

The hatted indices refer the orthonormal coordinates. Also, $\rho$ represents the total energy density and $T_{\mu\nu}$ are the components of the total energy–momentum tensor while $T$ is its trace.

For brane winding modes the total energy momentum tensor for a particular winding pattern can be shown to be a sum of dust-like energy momentum tensors with vanishing pressure coefficients wherever there is no wrapping and minus one wherever there is wrapping [1–4]. The energy density for any such conserved energy–momentum tensor would be

$$\rho^\alpha = \rho_0^\alpha \exp \left[ -m B + \sum_i (1 + \omega_i C_i) \right]. \quad (4)$$

with $\rho_0^\alpha > 0$ and as mentioned $w_i = -1$ for directions where there is a wrapped brane and $w_i = 0$ if there is no brane wrapped in that direction. Following [4] we call the $w_i = -1$’s, winding, and $w_i = 0$’s, transverse directions respectively. Note in particular that since branes only wrap around extra dimensions, observed space is transverse to those and the corresponding pressure coefficient vanishes: we have $T_{bb} = 0$.

Now, if stabilization ever happens the rest of the equations should remain compatible. Stabilization
would require all $\dot{C}_i$ and $\dot{C}_j$ to vanish and we therefore get the following relation (upon observing that $\sum_ip_i$ [RHS of $C_i$ equations] = 0)

$$\left(\frac{m-1}{d-2}\right)T = -\rho.$$  

(5)

This results in the following equations for $B$

$$\ddot{B} + m\dot{B}^2 = \left(\frac{1}{m-1}\right)\rho,$$  

(6a)

$$m(m-1)\dot{B}^2 = 2\rho.$$  

(6b)

Here $\rho = e^{-mb} \times \sigma$. The constant $\sigma$ depends on values of the stabilized scale factors $C_i(0)$ and energy density factors $\rho_i^0$. Eqs. (6) for $B$ are congruent only if $e^B \propto t^{2/m}$ which is the evolution of pressureless dust. This is to be expected since branes do not exert any pressure along the observed dimensions. The only remaining condition is

$$\sigma = 2e^{mb(0)}\left(\frac{m-1}{m}\right),$$  

(7)

which can be satisfied without problem since we still have the choice of $B(0)$.

Thus if we can somehow find a way to stabilize the extra dimensions, these solutions will not spoil the rest of the equations and we would be safe. From now on we will focus on the $C_i$ equations to study stabilization.

### 2.1. 2-fold decimation

In this section we would like to reproduce the results presented in [3,4] to clarify the formalism. We divide the space–time as $1 + m + p + q$, that is the extra dimensions are divided in a 2-fold partitioning. We also use the following winding scheme

$$(p)q \oplus p(q).$$

The above is meant to read that there is one brane wrapping along the $p$ directions alone and another one wrapping along the $q$ directions alone. The $C_i$ equations are therefore

$$-\frac{m+q-2}{d-2}\rho_0^p e^{-qC_q} + \frac{1+q}{d-2}\rho_0^q e^{-pC_p} = 0,$$  

(8a)

$$\frac{1+p}{d-2}\rho_0^p e^{-qC_q} - \frac{m+p-2}{d-2}\rho_0^q e^{-pC_p} = 0.$$  

(8b)

Which could be written as a matrix equation

$$\begin{bmatrix}
-(m+q-2)1+q \\
1+p-(m+p-2)
\end{bmatrix}
\begin{bmatrix}
X \\
Y
\end{bmatrix} = 0.$$  

(9)

Here we have defined $X = \rho_0^pe^{-qC_q}$ and $Y = \rho_0^q e^{-pC_p}$ and omitted the irrelevant factors. For a non-trivial solution we must require the determinant of the matrix in (9) to vanish. This quantity is $(m-3)(d-2)$, therefore the necessary requirement is $m = 3$. On the other hand the solutions must all be positive definite as evident from the definitions of $X$ and $Y$. With $m = 3$ the nullspace of the matrix in (9) is $(1,1)$, therefore there is stabilization and the necessary and sufficient condition is $m = 3$.

We could enlarge the winding scheme to the following

$$(p)q \oplus p(q) \oplus (pq),$$

and this would give in return

$$\begin{bmatrix}
-(m+q-2)1+q \\
1+p-(m+p-2)
\end{bmatrix}
\begin{bmatrix}
X \\
Y
\end{bmatrix} = (m-2)\rho_0^p q.$$  

(10)

This would result in $X = Y = -(m-2)/(m-3)\rho_0^p q$ and again a positive definite solution would not be possible. Therefore the winding mode $(pq)$ is forbidden for stabilization in this case.\(^2\)

### 2.2. 3-fold decimation

To get further acquainted with the formalism let us consider a partitioning of the form $1 + m + p + q + r$. And consider the following cascaded winding scheme

$$(pq)r \oplus p(qr) \oplus q(rp).$$

With this matter content the stabilization equations for extra dimensions read

$$\begin{bmatrix}
-(m+r-2)1+q+r-(m+q-2) \\
-(m+r-2)-(m+p-2)1+p+r \\
1+p+q-(m+p-2)-(m+q-2)
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} = 0.$$  

(11)

With $X = \rho_0^p q e^{-rC_r}$ and similar definitions for the others. The determinant of the matrix in (11) is

\(^2\) As will be clear from discussions below, $N = 2$ is a special case where the results of the remaining parts of this work is not valid.
This scheme will bring quadratic terms involving \( XY(pqr) \) similarly the set \( p \) will have equal winding densities when \( \rho_p \) transverse partitions the corresponding energy density will be \( \rho_{pqr} \). The solutions to this system would be

\[ X \cdot \text{tor} = 2m(d - 2), \]

which would not allow a non-trivial solution for integer \( m \). To be able to go around this we could add the \( (pq) \) winding mode so that the total winding scheme becomes

\[ (pq) \oplus p(qr) \oplus q(rp) \oplus (pqr), \]

for which the stabilization equations are,

\[
\begin{bmatrix}
-(m + r - 2)1 + q + r - (m + q - 2) \\
-(m + r - 2) - (m + p - 2)1 + p + r \\
1 + p + q - (m + p - 2) - (m + q - 2)
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
= (m - 2)\rho_0^{pq} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

The solutions to this system would be \( X = Y = Z = \rho_0^{pq}(m - 2)/(5 - 2m) \) which makes it impossible to have positive definite solutions for \( m \geq 2 \).

The only possibility left is to consider further winding modes around one part of the decimation. That is we now look for the following winding scheme

\[ (pq) \oplus p(qr) \oplus q(rp) \oplus (pqr) \oplus (p)qr \oplus p(qr) \oplus pq(r). \]

This scheme will bring quadratic terms involving \( XY, XZ \) and \( YZ \). So this system will in principle be rather complicated. Nevertheless we can devise a simpler way to proceed as follows.

The equations are invariant under the trivial permutations of quantities depending on \( p, q \) and \( r \). Therefore if a stabilizing solution exists generally, it should also exist for \( p = q = r \) and with this we should take all the corresponding coefficients of energy densities to be equal since the only parameter these can depend are the dimensionality of the space around which they wrap (and topology which we took to be the same from the outset). That is, the modes \( (pq)r, p(qr) \) and \( q(rp) \) will have equal winding densities when \( p = q = r \) and similarly the set \( (pq)r, p(q)r \) and \( pq(r) \) will have equal winding densities for \( p = q = r \) (and the mode \( (pqr) \) will have another strength). This is what we would like to call symmetric decimation and although it is not the general case it would give a hint on the general solution. That is, if after setting \( p = q = r \) the solution does not depend on \( p \) we can argue as follows.

Assume a non-symmetric solution exists and there is stabilization, then all the differences in the stabilization values of the scale factors of extra dimensions can be gauged away by rescaling the corresponding dimensions by appropriate amounts. Thus a symmetric solution hints strongly to a solution in the general case.

This argument simplifies matters to a considerable extent since we will have only one variable to deal with and the problem will transform to the study of positive definite solutions for the above equations for all positive \( \alpha \) and \( \beta \) unless \( m = 3 \). This actually does not depend on the existence of \( \beta \), provided the linear term in \( X \) is present.

Thus we have shown that in 3-fold decimation a democratic winding scheme and hence a democratic intersection scheme is stabilizing the extra dimensions with positive real scale factors provided the dimensionality of the observed space is constrained to be \( m = 3 \).

### 2.3. \( N \)-fold symmetric decimation

The formalism in the previous part can be generalized easily to an \( N \)-fold democratic decimation and following the arguments we presented we will consider symmetric decimation:

\[ 1 + m + p + p + p + \cdots \]

the winding scheme will consist of

\[ (p)pppp \ldots \oplus N \text{ sources} \oplus (pp)pppp \ldots \oplus N \text{ sources} \oplus (ppp)ppp \ldots \oplus N \text{ sources} \oplus \]

all \( N - 1 \) modes \( \vdots \) \( N \) sources \( \oplus \)

\[ (ppp \ldots) \] 1 source

---

\( ^3 \) Since for example the winding mode \( (pq)r \) will have two transverse partitions the corresponding energy density will be

\[ \rho_0^{pq} e^{-mB - qC} - rC, \]

which is proportional to \( XY \) apart from the factor \( e^{-mB} \) which will cancel out in the stabilization equations.
and this will lead to the following stabilization condition

\[ P_s(X) = \sum_{n=1}^{N-1} \alpha_n \xi_n X^n - \beta (m - 2) = 0 \]  

(14)

with

\[ \xi_n = (2N - n) - (N - n)m, \]

\[ \alpha_1 = 1, \]

\[ \alpha_i \geq 0, \]

\[ \beta \geq 0. \]  

(15a-d)

Here \( \alpha_i \) and \( \beta \) are related to the winding densities of the corresponding modes. Stabilization requires finding the solutions of (14) for all \( \alpha_i \) and \( \beta \) such that \( X > 0 \). For our purposes we do not need to know the most general solutions of polynomials of arbitrary order. We just need to find the condition on \( m \) such that there are positive roots. To study this we just need to count the sign changes in the polynomial starting from the highest order term, then Descartes’s sign rule tells us that the number of positive roots is either equal to the number of sign changes in the coefficients or less than it by a multiple of 2. The sign of the coefficients is determined by \( \xi_n \). Thus, if there will be a sign change after all it will only occur once starting with the lowest degree term and the sign changing term changes and moves towards the highest degree term with increasing \( m \).

It can be shown that there are no sign changes for \( m < 3 \) and \( m \geq N \) (the upper bound starts to operate for \( N \geq 3 \)) and that there is only one sign change in between, meaning that there is only one positive root for all values of \( \alpha_i \).

Again this is independent of the full winding mode \((p p p p \ldots)\), provided we have the term linear in \( X \), since the coefficient of the term is \((2 - m)\) which always starts to change sign with the linear term. Unfortunately this procedure does not fix \( m \) unambiguously but it certainly puts a strict lower bound of \( m \geq 3 \) and a decimation dependent upper bound of \( m \leq N \) which is enough to state that \( m \geq 3 \) is always sufficient to ensure stabilization although it is not necessary. One could turn the argument around and actually pick the decimation number to be \( N = 3 \) which is the smallest number of decimations for which the bound operates and this unambiguously fixes \( m \). Finally the fact that there is only one positive root hints at the possibility that the solution exists even for non-symmetric decimation.

To complete the argument we should remember that Eq. (7) be satisfied as well. With the parameters defined in (14) and (15) this will read

\[ \beta + N \sum_{i} \alpha_i X^i = 2e^{mB(0)} \left( \frac{m - 1}{m} \right), \]  

(16)

which can be satisfied without problem by an appropriate choice of \( B(0) \).

2.4. Stability of the equilibrium point

In this section we discuss that the stability point is in fact stable, that is all admissible (see below) initial data will converge to the stabilization point. We first show that the equilibrium point is linearly stable and then present an argument to show that stabilization is also achieved in the non-linear regime.

\[ \ddot{C} = -\dot{A} \dot{C} + \frac{1}{d + 2} e^{-mB} P_s(X = e^{-\delta C}). \]  

(17)

If we expand \( C \) around the equilibrium point such that we keep only the linear perturbations \( C = C_0 + \delta C \) we would get the following

\[ \delta \ddot{C} = -\dot{A} \dot{C} - \frac{P(X_0)}{d + 2} e^{-mB} X_0 P'_s(X_0), \]  

(18)

where \( P'_s(X_0) \) denotes derivative of \( P_s \) at the equilibrium point.

This is like a motion under two forces. The force which is proportional to \( \dot{A} \) can either be a driving or a friction force depending on the sign of \( \dot{A} \) but as clear from the equations of motions \( \text{Sign}(\dot{A}) \) is a constant of motion since \( \dot{A} \) is never allowed to vanish by (3a). In [1] it has been observed that when \( \text{Sign}(\dot{A}) < 0 \) one has a driving force and a singularity is reached in finite proper time. For \( \text{Sign}(\dot{A}) > 0 \) on the other hand one has a friction-like force, this already is a hint for the stability of the equation in the long term. The other force is like a linear force depending on the sign of the RHS of (18). This sign is unambiguously fixed by the requirements for stabilization. Let us remember that the coefficients of \( P_s(X) \) start being coming negative with increasing \( m \) starting from the lowest order term (in cluding the constant term), and we have shown
that there is a unique positive root of $P_t(X)$ if there is at least one sign change in the coefficients. Thus the requirement for a unique non-degenerate positive root also requires that the sign of the coefficient of the highest order term be positive, this would mean that the derivative of the stabilization polynomial at the equilibrium point is positive since the positive root is the largest root and the polynomial will either increase or decrease depending on the sign of the coefficient of the largest order term. Therefore the sign of the RHS of (18) is negative. The second force is an attractive linear force in the linear approximation. Hence the $C$ equations are linearly stable near the stabilization fixed point.

To try to understand what could happen in the non-linear regime let us remember again we have shown, given the conditions on $m$, that there is a unique positive root $X_0$ of $P_t(X)$. This would mean that we have $P_t(X) < 0$ for $0 < X < X_0$ and $P_t(X) > 0$ for $X > X_0$ since the coefficient of the highest order term is positive. We will confine ourselves to $\text{Sign}(\dot{A}) > 0$.

**Case A** Let us pick a point in the region $0 < X < X_0$ as our initial data point. Here $\dot{C}$ is initially negative if initially $\dot{C} > 0$ (i.e. getting larger $X$ getting smaller) and hence $\dot{C}$ will become negative at some time in the future (since $\dot{C}$ will never change sign until $\dot{C}$ becomes negative), when this transition happens we would have $C$ getting smaller and $X$ getting larger so this initial data eventually turns toward the equilibrium point $X_0$ with an initial data equivalent to $0 < X < X_0$ and $C < 0$ thus we should only consider this. In this case the sign of $\dot{C}$ depends on the interplay between the terms in (17). There are two cases: either the evolution of the system monotonically approaches $X_0$ with $\dot{C}$ approaching zero from above or passes this point with $\dot{C} > 0$ and becomes a problem of Case A above.

The considerations above are enough to argue that the point $X_0$ is an attractive fixed point; the solutions will converge to this point in the future.

**3. Conclusions**

In this work we have shown that in pure Einstein gravity the size of extra dimensions can be stabilized by winding modes alone in a particular winding system we called democratic winding scheme where intersections and windings of all possible kinds are allowed. This is an admirable result since during very early times in the cosmological evolution the extreme hot environment of the universe could somehow been incapable of choosing among winding schemes.

The mechanism presented here however should only work at early times due to the fact that it is hard to meet the fifth force constraint today since the brane mediated stabilization gives a tiny mass to the various shape factors.\footnote{Comment added after referee report.}

This model is incomplete in the sense that we have made the symmetric decimation choice and it is hard to infer directly what would happen for non-symmetric generalization. One possibly related problem is about the Branderberger–Vafa (BV) [24] mechanism which implies that all $p$-branes with $p > 2$ annihilated very early in the universe. To connect the BV construct to the model of this Letter we first realize that there are only four symmetric decimations of the six extra dimensions of string theory. These are 111111, 222, 33 and 6. Only the first two are allowed in the BV formalism to have non-trivial wrapping and unfortunately allowing only branes with $p \leq 2$ yields bounds for $m$ strictly greater than three. Perhaps a variant of the model we presented can circumvent this.

There could be interesting extensions of the democratic winding idea with increasing complexity. For example, it is fairly easy to see that adding momentum...
excitations along winding directions would possibly change things since the equations will be generally polynomials with non-integer powers. One could also try to enlarge the present system to dilaton gravity. Works along these lines are in progress.

References


5 This is expected because the pressure coefficient of momentum modes of a p-brane is 1/p in contrast to p independent winding pressure which is −1.