Fibonacciization and Multiparameter q-oscillators

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Abstract

Starting from Coon-Baker-Yu (CBY) q-oscillator, a two parameter, multidimensional q-oscillator is formulated. It is shown that this procedure, called Fibonacciization, can be extended to any number of parameters. The relation between CBY states and Fibonacci CBY states are also investigated.

Generalization of integers into q-basic numbers and generalization of quantum oscillators into q-oscillators [1,2,3] are related deformations. These deformations have been investigated in recent studies [4,6,7,8]. For example, oscillators with a spectrum given by basic numbers defined in symmetric form [9]

\[ [n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \]

(1)

where q is a real parameter, have been used in the construction of the SU_q(2) Lie algebra [2,3], q correction to Planck radiation and q statistics [10]. On the other hand, the Jackson basic number [12], given by

\[ [n] = \frac{q^n - 1}{q - 1}, \]

(2)

is used in the construction of SL_q(n) and SU_q(n) quantum [12,13]. This form also yields the energy spectrum of the multidimensional (nonquantum) U(n) invariant CBY q-oscillator [14]. In fact, the two parameter generalization of basic numbers,

\[ [n] = \frac{q_1^n - q_2^n}{q_1 - q_2}, \]

(3)
with $q_1, q_2$ real numbers or a complex conjugate pair, solves the Fibonacci difference equation [15]. Among multidimensional $q$-oscillators, the multidimensional $U_q(n)$ invariant Pusz-Woronowicz (PW) oscillator [16] can be extended to a two parameter oscillator. This is the covariant Fibonacci oscillator whose spectrum is given by (3).

On the other hand, the CBY oscillator is simpler to construct and is related to PW oscillator through the oscillator states [12]. We will now propose a two parameter version of the CBY oscillator using a similar procedure as in [15]. We will call it the Fibonacci-CBY oscillator.

The defining commutation relation of the CBY $q$-oscillator is given by

$$[a_i, a_j]_q = a_i a_j - qa_j a_i = \delta_{ij},$$

(4)

where $a$ and $a^\dagger$ are annihilation and creation operators, respectively. Using the fact that the right hand side of (4) is a scalar, we multiply (4) by $a^\dagger_j$ once from left, once from right and rearrange to obtain

$$a_i a^\dagger_j a^\dagger_k = qa^\dagger_j a_i a^\dagger_k - qa^\dagger_k a^\dagger_j a_i + a^\dagger_k a_i a^\dagger_j.$$  

(5)

We now introduce a new parameter $q_2$ by defining a new operator $b$ by

$$b_i = q_2^{N/2} a_i,$$

(6)

where $N$ is the number operator satisfying

$$a_i N = (N + 1) a_i.$$

Using this fact it follows that

$$b_i f(N) = f(N + 1) b_i$$

(7)

$$b_j^\dagger f(N) = f(N - 1) b_j^\dagger$$

for any analytic function $f(N)$.

It is possible to eliminate $q_2^{-3N/2}$ and find

$$b_j^\dagger b_j^\dagger_k = qq_1 b_j^\dagger b_j^\dagger_k - qq_2 b_j^\dagger b_j^\dagger_k + q_2 b_j^\dagger b_j^\dagger_k.$$  

(8)

If we set $q = q_1 / q_2$, we obtain a two parameter version of the CBY oscillator which we will call the fCBY oscillator with the “commutation” relation

$$b_i b_j^\dagger b_k^\dagger = q_1 b_i b_j^\dagger b_k^\dagger - q_1 q_2 b_k^\dagger b_j^\dagger b_i + q_2 b_k^\dagger b_i b_j^\dagger,$$

$$b_i |0> = 0,$$

$$b_i b_m^\dagger |0> = \delta_{im} |0>$$

(9)
For the above equation (9), we calculate the inner products of oscillator states defined through:

\[ |ijk\ldots> = b_j^\dagger b_k^\dagger \ldots >, \]

where \(|0>\) is the normalized ground state:

\[
<i|j> = \delta^i_j, \quad <i|j|m> = q_1^i \delta^i_m <j|k>, \quad <ijk|mn> = q_2^i \delta^i_m <j|k|n> + q_2^i \delta^i_n <j|k|m> + q_1^i \delta^i_p <j|km>.\]

The recursive property of the above definition permits us to define a tensor \(N\) from inner products of the fCBY q-oscillator states

\[
<i_1\ldots i_n|k_1\ldots k_n> \equiv N^{i_1i_2\ldots i_n}_{k_1k_2\ldots k_n} \equiv N^i_k(n, q), \quad (11)
\]

which has the recursion relation

\[
N^{i_1i_2i_3\ldots i_n}_{k_1k_2k_3\ldots k_n} = q_2^{n-1} N^{i_1i_2i_3\ldots i_n}_{k_1k_2k_3\ldots k_n} + q_2^{n-2} N^{i_2i_3\ldots i_n}_{k_1k_2k_3\ldots k_n} + q_1^{n-3} N^{i_1i_2i_3\ldots i_n}_{k_1k_2k_3k_4\ldots k_n} + \ldots + q_1^{n-1} N^{i_2i_3\ldots i_n}_{k_1k_2k_3\ldots k_{n-1}} \quad (12)
\]

If we consider the simplest form of the \(N\) tensor, \(N^i_j\), we see that it is simply the Kronecker delta, \(\delta^i_j\). Since for the case \(q_1 = -1\) or \(q_2 = -1\), where we find the usual \(n\)-dimensional Kronecker delta, we name \(N\) the two parameter generalized Kronecker delta.

It is also possible to find a relation between the \(N\) tensor of the CBY q-oscillator and the \(N\) tensor of the new fCBY oscillator. To accomplish this, we expand the \(N\) tensor completely and we see that for two and three particle cases we have

\[
N = q_2 N_{CBY}|_{q=q_1/q_2} \quad (13)
\]

where \(N\) denotes the sensor obtained from the inner products of CBY states [1,17]. For the generalization do a \(d\)-dimensional inner product case, one should note that in the above examples, \(N\) is each time divided by the highest power of \(q_2\) when it is expanded.
into its most simple form using (12). To find the expression for the highest power of $q_2$, we use the fact that the defining equation (9) is symmetric in powers of $q_1$ and $q_2$.

To obtain the highest power of $q_1$, in a $d$-dimensional inner product,

$$<i_1 i_2 \cdots i_d | j_1 j_2 \cdots j_d> = <b_{i_1} \cdots b_{i_d} a_{j_1} b_{j_2} \cdots b_{j_d} | >,$$

we couple each $b_{i_k}$ with the rightmost possible $b_{j_m}$ giving rise to a factor of $q_1$ each time it is permuted with a $b_{j_1}$. For example, $b_{i_1}$ will couple with $b_{j_d}^d$ picking up $(d-1)$ factors of $q_1$. If we denote the total power of $q_1$ by $m(d)$, it can be expressed by a summation

$$m(d) = \sum_{i=1}^{d} (d - i) = \frac{d(d-1)}{2}.$$  \hspace{1cm} (14)

Using the symmetry of $q_1$ and $q_2$, we find that

$$N = q_2^{m(d)} N_{\text{CBY}}|_{q \equiv q_1/q_2}.$$  \hspace{1cm} (15)

The defining relation for this new oscillator (9) can also be written in the form of q-commutator notation in (4):

$$[[b_{i_1}, b_{j_1}], b_{k_1}]_{q_2} = 0.$$  \hspace{1cm} (16)

We can generalize this equation by writing

$$[[[b_{i_1}, b_{j_1}], b_{k_1}], b_{l_1}]_{q_2}, b_{m_1}]_{q_3} = 0.$$  \hspace{1cm} (17)

Note that (16) trivially solves (17). Further generalizing this equation by introducing new parameters is straightforward. One can construct the $n$-parameter version of ICBY q-oscillator as

$$[[[\cdots[[b_{i_1}, b_{j_1}], b_{k_1}], b_{l_1}], \cdots]_{q_n} = 0.$$  \hspace{1cm} (18)

By setting $i = j = k = \cdots = z$, it is possible to obtain the one dimensional version of the above new q-oscillator. If this equation is written as a difference equation, the associated $n$-parameter basic number can be calculated. One should note that, for $n$ parameters, the difference equation which determines the spectrum is of order $n$ and therefore the number of initial conditions is also $n$. Different choices of initial conditions lead to different results. For example, if one sets the first $(n-1)$ basic numbers to be the integers up to $(n-1)$ as the initial conditions, then it is possible to think of $n$, as the point above which the $q$ effects are to be observed. Another interesting case can be an $(n-1)$-fold degenerate ground state and a unique first excited state.

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References