Electric Dipole Radiation

Physics 202

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Notes compiled by Alpar Sevgen

In these notes electric dipole radiation is discussed.

I found the following references very useful:[1] and [2].

1. Electric Dipole Radiation

Accelerating electric charges emit electromagnetic radiation. These radiation fields have the following characteristics

- Radiation fields $\vec{E}_R$, $\vec{B}_R$ behave as $\frac{1}{r}$ at large distances $r \to \infty$.

- $\vec{E}_R$, $c\,\vec{B}_R$ and $\hat{\mathbf{r}}$ (direction of propagation) form a right-handed triad, and thus $\vec{E}_{R,p}$, $c\,\vec{B}_{R,p}$ are *transverse* fields.

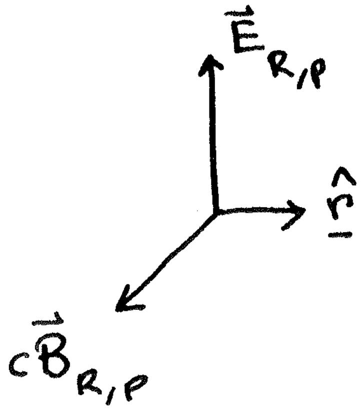


Figure 1: $\vec{E}_{R,p}$, $c\,\vec{B}_{R,p}$ are *transverse* fields

- $\vec{E}_R$ and $\vec{B}_R$ are proportional to the acceleration $\vec{a}$ of the charges.

- $\vec{E}_R$ and $\vec{B}_R$ depend on the retarded time:

$$t' = t - \frac{R}{c} \qquad \text{where } R = |\vec{r} - \vec{r}'|\,, \tag{1}$$

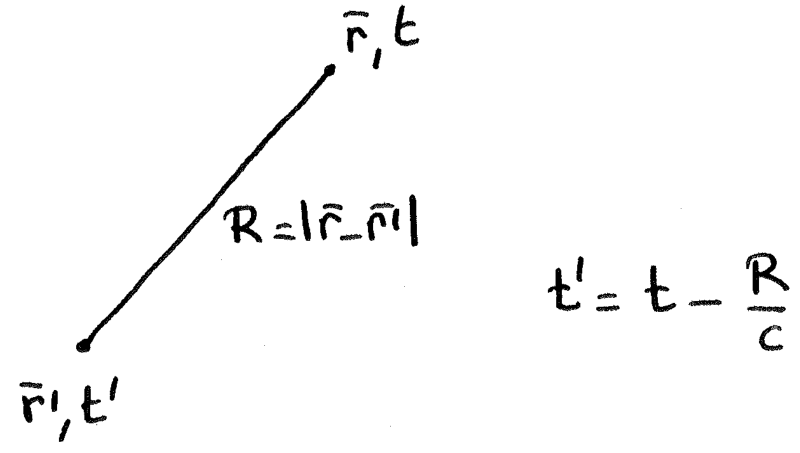


Figure 2: The radiation fields $\vec{E}_R$, $\vec{B}_R$ observed at $(\vec{r}, t)$ depend on the retarded source time $t' = t - R/c$, Eq.(1).

We now specialize to the case where the wavelength λ of the radiation is much larger than the maximum linear dimension D of the source region, that is $\lambda >> D$, and that the motions of charges are nonrelativistic $\frac{\omega D}{c} << 1$, and observation point is far away $r >> \lambda$. In that case electric dipole approximation to the radiated field is very good, and we simply quote the result for a dipole $\vec{p}(t)$ at the origin,

$$\vec{E}^{E1}_{R,phys}(\vec{r}, t) = -\frac{1}{4\pi\epsilon_o c^2}\frac{\ddot{\vec{p}}_{\perp,phys}\,(t - r/c)}{r}, \tag{2}$$

where $\perp$ indicates perpendicular to $\hat{\mathbf{r}}$. In many important cases $\vec{p}(t)$ has harmonic time dependence

$$\vec{p}_{physical} = Re\,\vec{p}(t) = Re\,\vec{p}_o e^{-i\omega t}\,, \tag{3}$$

where $\vec{p}_o$ may be a complex valued vector. Then we set

$$\vec{E}_{R,physical} = Re\vec{E}_R \tag{4}$$

and thus we can now use the complex valued fields

$$\vec{E}^{E1}_R(\vec{r}, t) = -\frac{1}{4\pi\epsilon_o c^2}\frac{\ddot{\vec{p}}_{\perp}\,(t - r/c)}{r}. \tag{5}$$

Let us now discuss the electric dipole

$$\begin{aligned} \vec{p}(t) &= \vec{p}_o e^{-i\omega t} \\ \vec{p}(t - r/c) &= \vec{p}_o e^{-i\omega(t-r/c)} \\ &= \vec{p}_o e^{-i\omega t} e^{ikr} \quad \text{where } k = \tfrac{\omega}{c} \end{aligned} \tag{6}$$

and the perpendicular component reads

$$\vec{p}_\perp(t - r/c) = \vec{p}_{o\perp}\, e^{i(kr-\omega t)}. \tag{7}$$

Acceleration of the dipole is easily computed

$$\ddot{\vec{p}}_\perp\,(t - r/c) = -\omega^2 \vec{p}_{o\perp}\, e^{i(kr-\omega t)}. \tag{8}$$

So that, for a harmonically varying electric dipole at the origin Eq.(5) gives

$$\begin{aligned} \vec{E}_R^{E1}(\vec{r}, t) &= \left(\frac{1}{4\pi\epsilon_o}\right) k^2 \vec{p}_{o\perp} \left\{\frac{e^{i(kr-\omega t}}{r}\right\} \\ &= \vec{E}_{oR}^{E1}(\vec{r})\, e^{-i\omega t} \end{aligned} \tag{9}$$

This is such an important formula that we write it again with a box around it. The term in the curly brackets is *outgoing spherical wave* with speed c.

$$\boxed{\vec{E}_R^{E1}(\vec{r}, t) = \left(\frac{1}{4\pi\epsilon_o}\right) k^2 \vec{p}_{o\perp} \left\{\frac{e^{i(kr-\omega t}}{r}\right\}}$$

The angle dependence of electric field comes from the perpendicular component of the amplitude of the dipole amplitude

$$\vec{p}_{o\perp} = \vec{p}_o - \hat{\mathbf{r}}\ \hat{\mathbf{r}}.\vec{p}_o\,, \tag{10}$$

where

$$\hat{\mathbf{r}} = sin\theta\, cos\phi\, \hat{\mathbf{x}} + sin\theta\, sin\phi\, \hat{\mathbf{y}} + cos\theta\, \hat{\mathbf{z}}\,. \tag{11}$$

Note that the vector $\vec{p}_{o\perp}$ is orthogonal to the direction vector $\hat{\mathbf{r}}$, that is $\hat{\mathbf{r}}.\vec{p}_{o\perp} = 0$.

Since the fields are transverse as mentioned in the general properties of the radiation, we can obtain the magnetic field, in the electric dipole approximation as

$$c\vec{B}_R^{E1} = \hat{\mathbf{r}} \times \vec{E}_R^{E1}\,, \tag{12}$$

or explicitly as,

$$\begin{aligned} \vec{B}_R^{E1}(\vec{r}, t) &= \left(\frac{1}{4\pi\epsilon_o}\right) \frac{k^2}{c} \hat{\mathbf{r}} \times \vec{p}_{o\perp} \left\{\frac{e^{i(kr-\omega t}}{r}\right\} \\ &= \vec{B}_{oR}^{E1}(\vec{r})\, e^{-i\omega t} \end{aligned} \tag{13}$$

Radiated power (current of energy) passing through an infinitesimal spherical area is

$$dP = \vec{S}.\left(\hat{\mathbf{r}}\, r^2\, d\Omega\right) \;, \tag{14}$$

(compare this to the current of charge passing through an infinitesimal area $dI = \vec{j}.\hat{\mathbf{n}}\; da$). Power radiated per unit solid angle is then

$$\frac{dP}{d\Omega} = \vec{S}.\left(\hat{\mathbf{r}}\, r^2\right) \;, \tag{15}$$

where $\vec{S}$ is the Poynting vector

$$\vec{S} = \epsilon_o c^2 \left(\vec{E}^{E1}_{R,p} \times \vec{B}^{E1}_{R,p}\right) \;, \tag{16}$$

where the fields are *real.* We are not interested in rapidly fluctuating instantaneous values, but in smoothly varying *time-averaged* quantities. Averaging is done over the period of oscillation of the source. Thus the average power radiated per unit solid angle is defined as

$$\frac{d\overline{P}}{d\Omega} = \frac{1}{T}\int_0^T dt\, \frac{dP}{d\Omega}\,, \tag{17}$$

where $\overline{P}$ indicates power averaged. Because of averaging, we can now use the complex form of the fields. We will need a vector relation among vectors

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{b}\;\; \vec{a}.\vec{c} - \vec{a}\;\; \vec{b}.\vec{c} \tag{18}$$

We now compute the time averaged power radiated (we also drop the superscript $E1$ for simplicity of notation),

$$\begin{aligned} \frac{d\overline{P}}{d\Omega} &= r^2\hat{\mathbf{r}}. < \vec{S} > \\ &= \epsilon_o c^2\; r^2\hat{\mathbf{r}}.\frac{1}{2}\; Re\left(\vec{E}_{oR} \times \vec{B}^*_{oR}\right) \end{aligned} \tag{19}$$

We now use the forms of $\vec{E}_{oR}$ and $\vec{B}_{oR}$ given in Eqs.(9 and 12) in (19)

$$\begin{aligned} \left(\vec{E}_{oR} \times \vec{B}^*_{oR}\right) &= \vec{E}_{oR} \times (\hat{\mathbf{r}} \times \vec{E}^*_{oR}/c) \\ &= \frac{1}{c}\left(\vec{E}^*_{oR} \times \hat{\mathbf{r}}\right) \times \vec{E}_{oR} = \frac{1}{c}\hat{\mathbf{r}}|\vec{E}_{oR}|^2 - \frac{1}{c}\vec{E}^*_{oR}\; \hat{\mathbf{r}}.\vec{E}_{oR} \\ &= \frac{1}{c}\; \hat{\mathbf{r}}\; |\vec{E}_{oR}|^2\,, \end{aligned} \tag{20}$$

where in the middle line in (20) we used the vector relation (18) and the fact that $\vec{E}_{oR}$ is perpendicular to $\hat{\mathbf{r}}$ so that $\hat{\mathbf{r}}.\vec{E}_{oR} = 0$. We put all these back into the formula for the time averaged power radiated per unit solid angle, and obtain

$$\frac{d\overline{P}}{d\Omega} = \left(\frac{1}{4\pi\epsilon_o}\right) \frac{ck^4 |\vec{p}_{o\perp}|^2}{8\pi}, \tag{21}$$

where we use Eq.(10) and

$$|\vec{p}_{o\perp}|^2 = \vec{p}^{\,*}_{o\perp} . \vec{p}_{o\perp} = |\vec{p}_o|^2 - |\hat{\mathbf{r}}.\vec{p}_o|^2 . \tag{22}$$

The total average power radiated is obtained by integrating over all angles

$$\overline{P} = \int d\Omega \, \frac{d\overline{P}}{d\Omega} \tag{23}$$

In Eq.(23) integration over the solid angle Ω means the integration over the angles of the unit vector $\hat{\mathbf{r}}$ given in (11), with F a function of angles

$$\int d\Omega \; F(\Omega) = \int d\hat{\mathbf{r}} \; F(\hat{\mathbf{r}}) = \int_o^{2\pi} d\phi \int_0^{\pi} d\theta sin\theta \; F(\theta, \phi) \tag{24}$$

And if $F(\theta, \phi)$ is a function of $\mu = cos\theta$ then by a change of variable (24) becomes

$$\int d\Omega \; F(\Omega) = \int_o^{2\pi} d\phi \int_{-1}^{+1} d\mu \; F(\mu, \phi) \tag{25}$$

Exercise 1. Electric dipole not at the origin

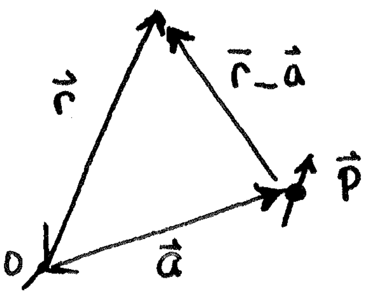


Figure 3: A single dipole at the position $\vec{a}$

Suppose an electric dipole is at the position $\vec{r}_d = \vec{a}$ where $a << r$. You will now compute and find out what the fields are and how our formulas change. The exact radiation electric field (in the dipole approximation) is

$$\vec{E}_R(\vec{r},t) = \left(\frac{1}{4\pi\epsilon_o}\right) k^2 \vec{p}_{o\perp} \left\{ \frac{e^{i(k|\vec{r}-\vec{a}|-\omega t)}}{|\vec{r}-\vec{a}|} \right\} \tag{26}$$

where

$$\vec{p}_{o\perp} = \vec{p}_o - \hat{\mathbf{r}}_1 \; \hat{\mathbf{r}}_1.\vec{p}_o \quad \text{with } \hat{\mathbf{r}}_1 = \frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|} \tag{27}$$

The exact expression for the electric field of a displaced dipole (26) can be simplified quite a bit because $r >> a$, that is the field is evaluated at very large distances. In the following you will work on this simplification.

1. Show that, if $\vec{r}_1 = \vec{r} - \vec{a}$, then

$$r_1 = |\vec{r}-\vec{a}| = \sqrt{\vec{r}_1.\vec{r}_1} \approx r\left(1 - \frac{2\,\hat{\mathbf{r}}.\vec{a}}{r}\right)^{1/2} \approx r - \hat{\mathbf{r}}.\vec{a} \tag{28}$$

2. Show that

$$\frac{1}{|\vec{r}-\vec{a}|} = \frac{1}{r}\left(1 + \mathcal{O}\left(\frac{a}{r}\right)\right) \approx \frac{1}{r} \tag{29}$$

3. Show that

$$\begin{aligned} \frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|} &= (\vec{r}-\vec{a})\frac{1}{r}\left(1+\mathcal{O}\left(\frac{a}{r}\right)\right) \\ &= \hat{\mathbf{r}} + \mathcal{O}\left(\frac{a}{r}\right) \approx \hat{\mathbf{r}} \end{aligned} \tag{30}$$

4. Show that

$$e^{ik|\vec{r}-\vec{a}|} \approx e^{ikr} e^{-i\vec{k}.\vec{a}} \quad \text{where } \vec{k} = k\hat{\mathbf{r}}. \tag{31}$$

5. Thus the electric field of an electric dipole displaced from the origin differs from the electric field of the same dipole at the origin by a pure phase

$$\vec{E}(\vec{a}) = e^{-i\vec{k}.\vec{a}}\,\vec{E}(0). \tag{32}$$

Note that in Eq.(32) , both fields are at the radiation zone ($r \to \infty$), and we only indicated the dependence on the dipole position.

6. Show that the magnetic field is affected by the same phase,

$$\vec{B}(\vec{a}) = e^{-i\vec{k}.\vec{a}}\,\vec{\boldsymbol{B}}(0). \tag{33}$$

7. Show that the angular distribution of the radiated average power does not change.

What did we learn from this important exercise?

1. If the electric dipole is displaced from the origin, the electric field acquires an overall phase. This does not affect the power distribution. That an overall phase will not affect the power distribution is clear from Eq.(20), in taking the absolute square, overall pure phase disappears.

2. It is clear that when there are two or more dipoles, each dipole field gets a phase depending on the position of the dipole, and thus when we add these fields to get the total field in the radiation zone, those phases will not be the same, and will be important.

$$\begin{aligned} \vec{E}_{Total}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_N) = e^{-i\vec{k}.\vec{a}_1}\ \vec{E}_1(0) + e^{-i\vec{k}.\vec{a}_2}\ \vec{E}_2(0) \\ + \ldots + e^{-i\vec{k}.\vec{a}_N}\ \vec{E}_N(0)\ , \end{aligned} \tag{34}$$

   and the electric field of each dipole $\vec{E}_j(0)$ differ only in the polarization vector $\vec{p}_{oj\,,\perp}$. Because different dipole fields have different phases, the total electric field is not multiplied only by an overall phase, and thus average radiated power per unit solid angle is certainly influenced by these phases.

End of Exercise 1

Exercise 2. Dipole array

There are N electric dipoles oscillating harmonically with the same angular frequency $\omega = ck$.

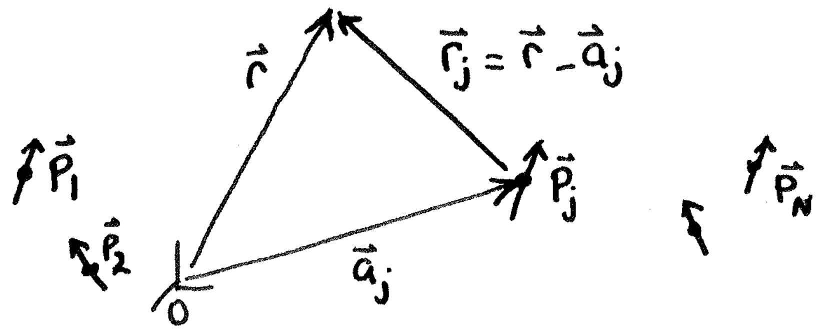


Figure 4: A dipole array of N dipoles

1. Show that the total electric field in the radiation zone is given as

$$\begin{aligned}\vec{E}_{Total}(\vec{r},t) =& \vec{E}_1(\vec{r},t) + \vec{E}_2(\vec{r},t) + \ldots + \vec{E}_N(\vec{r},t) \\ =& \left(\frac{1}{4\pi\epsilon_o}\right) k^2 e^{-i\vec{k}.\vec{a}_1} \vec{P}^{new}_{o\perp} \left\{\frac{e^{i(kr-\omega t)}}{r}\right\}\end{aligned} \tag{35}$$

$$\text{where} \quad \vec{P}^{new}_{o\perp} = \vec{p}_{o1,\perp} + e^{-i\vec{k}.(\vec{a}_2-\vec{a}_1)}\vec{p}_{o2,\perp} \\ e^{-i\vec{k}.(\vec{a}_3-\vec{a}_1)}\vec{p}_{o3,\perp} + \ldots + e^{-i\vec{k}.(\vec{a}_N-\vec{a}_1)}\vec{p}_{oN,\perp}$$

2. Since $\vec{r}_j = \vec{r} - \vec{a}_j$, show for example that

$$e^{ik(r_2-r_1)} \approx e^{-i\vec{k}.(\vec{a}_2-\vec{a}_1)} \tag{36}$$

   where

$$r_2 - r_1 \approx -(\vec{a}_2 - \vec{a}_1).\hat{\mathbf{r}} = \hat{\mathbf{r}}.(\vec{r}_2 - \vec{r}_1) \tag{37}$$

   Thus, dipoles are being added together with their phase differences due to the *path differences* in $\vec{P}^{new}_{o\perp}$. Note that if there are phase differences at the sources, they are included in $\vec{p}_{oj}$ terms.

3. Compare the forms of the electric fields in Eqs.(9 and 35). We see that the only difference is that $\vec{p}_{o\perp} \to e^{-i\vec{k}.\vec{a}_1}\vec{P}^{new}_{o\perp}$. Thus we can immediately obtain the angular distribution of the average power from an arbitrary dipole array by comparing with Eq(21),

$$\frac{d\overline{P}}{d\Omega} = \left(\frac{1}{4\pi\epsilon_o}\right) \frac{ck^4|\vec{P}^{new}_{o\perp}|^2}{8\pi}, \tag{38}$$

   where $\vec{P}^{new}_{o}$ is defined by Eq.(35), and

$$|\vec{P}^{new}_{o\perp}|^2 = |\vec{P}^{new}_{o}|^2 - |\hat{\mathbf{r}}.\vec{P}^{new}_{o}|^2 \tag{39}$$

4. Total power has the same form again

$$\overline{P} = \int d\Omega \, \frac{d\overline{P}}{d\Omega}. \tag{40}$$

End of Exercise 2

Various applications of radiations from dipole arrays involve using just Eqs.(35 and 38).

2. Examples

2.1 Dipole at the origin oriented along the z axis

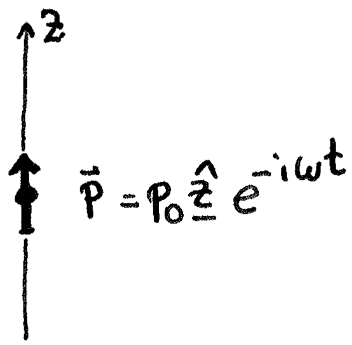


Figure 5: Dipole at origin oriented along z axis

In this case $\vec{p}_p = p_o \, \hat{\mathbf{z}} \, cos\omega t = Re \, p_o \hat{\mathbf{z}} e^{-i\omega t}$. So that $\vec{p}_o = p_o \hat{\mathbf{z}}$. The average power radiated was given in Eq(17). We have to compute $|\vec{p}_{o\perp}|^2$,

$$\begin{aligned} |\vec{p}_{o\perp}|^2 &= |\vec{p}_o|^2 - |\hat{\mathbf{r}}.\vec{p}_o|^2 \\ &= |p_o|^2(1 - cos^2\theta) = |p_o|^2 sin^2\theta \end{aligned} \tag{41}$$

So that average power radiated per unit solid angle is then

$$\frac{d\overline{P}}{d\Omega} = \left(\frac{1}{4\pi\epsilon_o}\right) \frac{ck^4 |p_o|^2}{8\pi} sin^2\theta, \tag{42}$$

and the angular distribution of radiation is shown in Figure(6) below,

To get the total power we have to integrate over all angles, that is over the directions of $\hat{\mathbf{r}}$, writing (42) as $\alpha \; sin^2\theta$, and inserting α explicitly at the end,

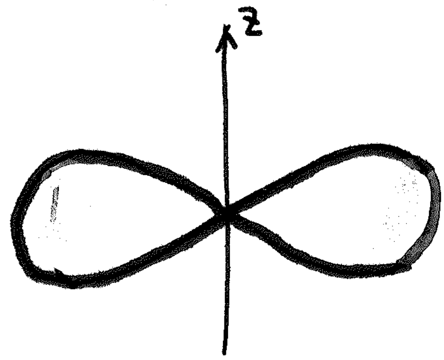


Figure 6: Angular distrubition for source in Fig.(5)

see Eqs.(24 and 25)

$$
\begin{aligned}
\overline{P} =& \int d\Omega \, \frac{d\overline{P}}{d\Omega} \\
=&\alpha \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, sin\theta \, sin^2\theta \\
=&\alpha \, 2\pi \int_{-1}^{+1} d\mu(1-\mu^2) \\
=&\alpha \, 2\pi \, 4/3 \\
=&\left(\frac{1}{4\pi\epsilon_o}\right) \frac{ck^4|p_o|^2}{3}
\end{aligned} \tag{43}
$$

2.2 Dipole at the origin rotating in the x, y plane

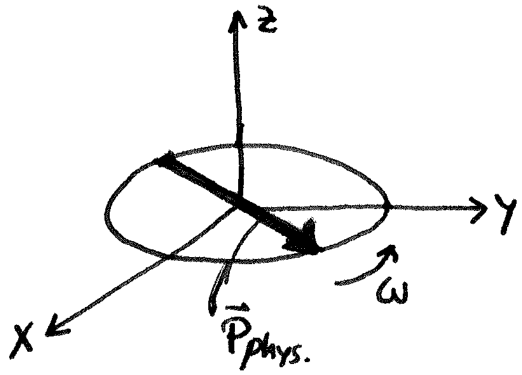


Figure 7: Dipole at the origin rotating in the x, y plane

The expression for the real physical dipole must be brought to the complex form $\vec{p}_o e^{-i\omega t}$,

$$\begin{aligned}\vec{p}_p(t) =&|p_o|\left(\hat{\mathbf{x}}cos(\omega t+\delta)+\hat{\mathbf{y}}sin(\omega t+\delta)\right)\\ =&Re\,\vec{p}_o e^{-i\omega t}\\ \vec{p}_o =&|p_o|e^{-i\delta}\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right).\end{aligned} \tag{44}$$

The electric field is given as in Eq.(9), and the time averaged power radiated per unit solid angle is given by Eqs.(21 and 22),

$$\vec{E}_R^{E1}(\vec{r},t) = \left(\frac{1}{4\pi\epsilon_o}\right) k^2|p_o|e^{-i\delta}\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right)_\perp \left\{\frac{e^{i(kr-\omega t)}}{r}\right\} \tag{45}$$

Exercise 3.

1. What is the polarization in the $+\hat{\mathbf{z}}$ direction, that is $\hat{\mathbf{r}} = \hat{\mathbf{z}}$? (Ans: right circular)
2. polarization in the $-\hat{\mathbf{z}}$ direction?
3. polarization in the $\hat{\mathbf{x}}$ direction? (Ans: Linear , parallel to y axis)
4. polarization in the $\hat{\mathbf{y}}$ direction? (Ans: Linear , parallel to x axis)
5. polarization in the general $\hat{\mathbf{r}}$ direction, not parallel to any of the axes?

End of Exercise 3

In order to compute the power radiated per unit solid angle we must compute

$$\begin{aligned}|\vec{p}_{o\perp}|^2 =&|\vec{p}_o|^2 - |\hat{\mathbf{r}}.\vec{p}_o|^2\\ |\vec{p}_o|^2 =&|p_o|^2\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right).\left(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\right) = 2|p_o|^2\end{aligned} \tag{46}$$

and we continue to compute the remaining terms

$$\begin{aligned}\hat{\mathbf{r}}.\vec{p}_o =&p_o(sin\theta\,cos\phi\; + i\,sin\theta\,sin\phi)\\ =&p_o\,sin\theta\,e^{i\phi}\end{aligned} \tag{47}$$

So that finally

$$|\vec{p}_{o\perp}|^2 = |p_o|^2(2-sin^2\theta) = |p_o|^2(1+cos^2\theta). \tag{48}$$

We can write the power radiated per unit solid angle now

$$\frac{d\overline{P}}{d\Omega} = \left(\frac{1}{4\pi\epsilon_o}\right)\frac{ck^4|p_o|^2}{8\pi}\left(1+cos^2\theta\right), \tag{49}$$

And the total power radiated is just the angular integral of (49). It is computed easily (24 and 25), lumping the constants into α,

$$\begin{aligned}\overline{P} &= \int d\Omega\, \frac{d\overline{P}}{d\Omega} = \alpha(4\pi + 2\pi \int_{-1}^{+1} d\mu\, \mu^2) \\ &= \left(\frac{1}{4\pi\epsilon_o}\right) \frac{ck^4|p_o|^2}{3} \times 2\,.\end{aligned} \tag{50}$$

It is interesting to note that the total power radiated by a rotating dipole (50) is twice the power radiated by a linear dipole (43). Why is that so? It is because a rotating dipole is equivalent to two linear dipoles oscillating with a $\pi/2$ phase between them. Show this.

2.3 Two dipoles along z axis, oscillating out of phase

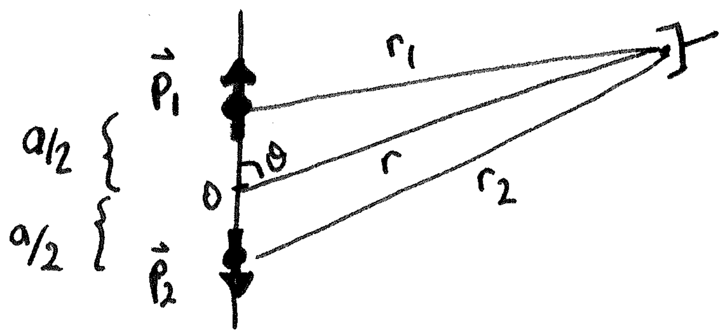


Figure 8: Two dipoles along z axis, oscillating out of phase

The dipoles and their positions are

$$\vec{p}_1 = p_o\hat{\mathbf{z}}e^{-i\omega t} \qquad \vec{a}_1 = \frac{a}{2}\hat{\mathbf{z}} \tag{51}$$

$$\vec{p}_2 = -p_o\hat{\mathbf{z}}e^{-i\omega t} \qquad \vec{a}_2 = -\frac{a}{2}\hat{\mathbf{z}} \tag{52}$$

The electric field is given by (35) and

$$\begin{aligned}\vec{P}^{new}_{o\perp} =&\vec{p}_{o1,\perp} + e^{-i\vec{k}.(\vec{a}_2-\vec{a}_1)}\vec{p}_{o2,\perp} \\ =&p_o\hat{\mathbf{z}}_\perp(1 - e^{-ik\hat{\mathbf{r}}.(-a\hat{\mathbf{z}})}) \\ =&p_o e^{i\frac{akcos\,\theta}{2}}(-2i)sin\left(\frac{akcos\,\theta}{2}\right)\hat{\mathbf{z}}_\perp\end{aligned} \tag{53}$$

We note that $\hat{\mathbf{z}}_\perp . \hat{\mathbf{z}}_\perp = 1 - cos^2\theta = sin^2\theta$ so that the time averaged power per unit solid angle becomes

$$\frac{d\overline{P}}{d\Omega} = \left(\frac{1}{4\pi\epsilon_o}\right) \frac{ck^4|p_o|^2}{8\pi} \, sin^2\theta \left(4sin^2\left(\frac{akcos\,\theta}{2}\right)\right) \tag{54}$$

Since $ak << 1$ and since $sinx \approx x$ as $x << 1$, to a good approximation Eq.(54) simplifies

$$\frac{d\overline{P}}{d\Omega} = \left(\frac{1}{4\pi\epsilon_o}\right) \frac{ck^4|p_o|^2}{8\pi} \, (ak)^2 \, sin^2\theta \, cos^2\theta \tag{55}$$

The radiation pattern is that of a quadrupole

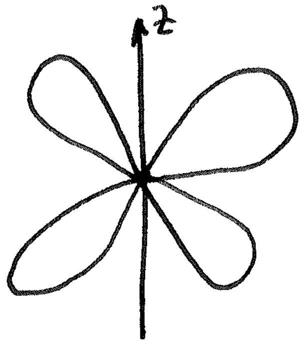


Figure 9: Quadrupole radiation

Total power is easy to find now (see (23, 24 and 25)).

$$\begin{aligned} \overline{P} &= \int d\Omega \, \frac{d\overline{P}}{d\Omega} = \alpha 2\pi \int_{-1}^{+1} d\mu \, \mu^2(1 - \mu^2) \\ &= \left(\frac{1}{4\pi\epsilon_o}\right) \frac{ck^4|p_o|^2}{3} \times \frac{(ak)^2}{5} = \overline{P}_1 \times \frac{(ak)^2}{5} \end{aligned} \tag{56}$$

where $\overline{P}_1$ is the total power radiated by a linear dipole Eq.(43). Note that this total power radiated is much less than power radiated by a single dipole, because the fields are largely *cancelling* each other.

2.4 Two in phase dipoles paralel to z axis

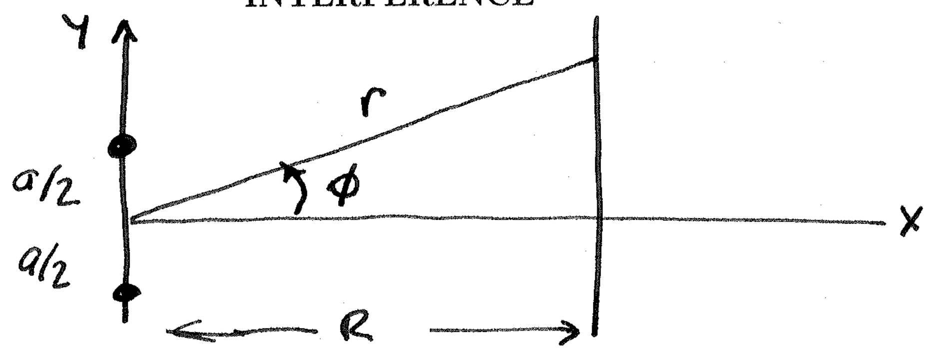


Figure 10: Two in phase dipoles parallel to z axis

The dipoles and their positions are

$$\vec{p}_1 = p_o \hat{\mathbf{z}} e^{-i\omega t} \qquad \vec{a_1} = \frac{a}{2}\hat{\mathbf{y}} \tag{57}$$

$$\vec{p}_2 = -p_o \hat{\mathbf{z}} e^{-i\omega t} \qquad \vec{a_2} = -\frac{a}{2}\hat{\mathbf{y}} \tag{58}$$

The electric field is given by (35) and

$$\begin{aligned} \vec{P}^{new}_{o\perp} &= \vec{p}_{o1,\perp} + e^{-i\vec{k}.(\vec{a}_2 - \vec{a}_1)} \vec{p}_{o2,\perp} \\ &= p_o \hat{\mathbf{z}}_\perp (1 + e^{-ik\hat{\mathbf{r}}.(-a\hat{\mathbf{y}})}) \\ &= p_o e^{i\frac{aksin\,\theta\,sin\,\phi}{2}}\, 2\cos\left(\frac{aksin\,\theta\,sin\,\phi}{2}\right)\hat{\mathbf{z}}_\perp \end{aligned} \tag{59}$$

Let us agree to detect the radiation along a line parallel to the y-axis as shown in the figure, so $\theta = \pi/2$, and $\hat{\mathbf{z}}_\perp = \hat{\mathbf{z}}$. So, the time averaged power per unit solid angle becomes

$$\frac{d\overline{P}}{d\Omega} = \left(\frac{1}{4\pi\epsilon_o}\right)\frac{ck^4|p_o|^2}{8\pi}\,4\left(cos^2\left(\frac{aksin\,\phi}{2}\right)\right) \tag{60}$$

It is better to consider the time averaged intensity of the radiation for this geometry. Intensity is defined as the power per unit area, so

$$\begin{aligned} I(\hat{\mathbf{r}}) &= <\vec{S}>.\hat{\mathbf{r}} = \frac{1}{r^2}\frac{d\overline{P}}{d\Omega} \\ &= 4\,I_o\,cos^2\left(\frac{aksin\,\phi}{2}\right), \end{aligned} \tag{61}$$

where I_o is the intensity due to a single dipole,

$$r^2 I_o \approx R^2 I_o = \frac{1}{4\pi\epsilon_o} \frac{ck^4|p_o|^2}{8\pi}. \tag{62}$$

Equation(61) shows that intensity varies between zero, and four times the single dipole value. Clearly we see the effect of *interference.* Waves from the two dipoles are sometimes adding, sometimes subtracting, giving rise to *constructive* or *destructive* interference.

In order to simplify drawing the interference pattern, however, let us make a small angle approximation

$$sin\phi \approx tan\phi = \frac{h}{R} \tag{63}$$

where h is the position of observation. If we also define the characteristic length h_o as

$$h_o = \frac{\lambda R}{a} \tag{64}$$

then we can write the intensity formula as

$$I(h) = 4\,I_o\,cos^2\left(\frac{\pi h}{h_o}\right), \tag{65}$$

so that, the positions of maxima are $h_{max} = nh_o$ with $n = 0, \pm1, \pm2\ldots$ and minima are $h_{min} = (n + 1/2)h_o$ as shown in the figure below.

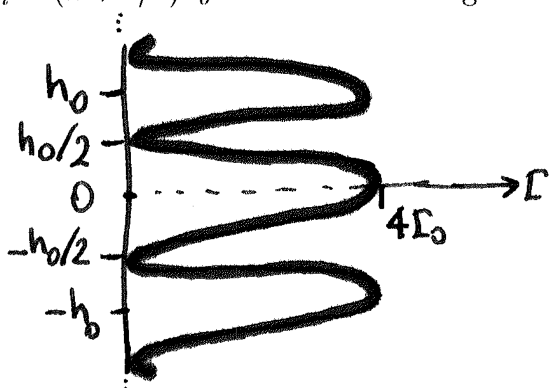


Figure 11: Interference pattern

Note that if we had not made a small angle approximation, the intensity is given by

$$I = 4I_o cos^2\left(\frac{\pi a}{\lambda}\,sin\phi\right) \tag{66}$$

And $I = 4I_o$ if $asin\phi_{max} = n\lambda$ and $I = 0$ if $asin\phi_{min} = (n + 1/2)\lambda$. That is when the path difference is a multiple of the wavelength there is a maximum, and when the path difference equals full wavelengths plus a half-wavelength intensity vanishes.

2 .5 N in phase dipoles paralel to z axis

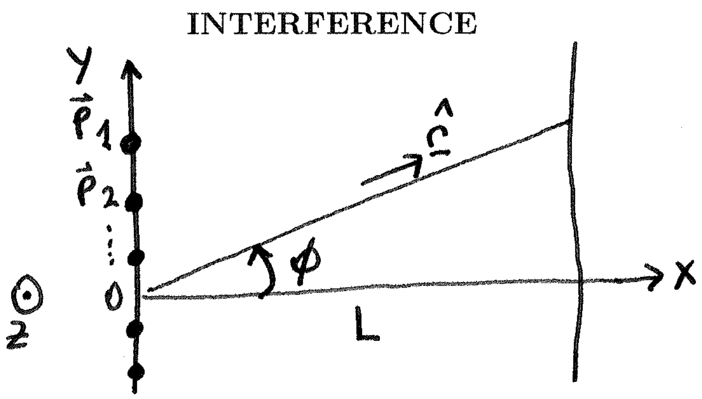


Figure 12: N in phase dipoles parallel to z axis

The dipoles and their positions are

$$\begin{aligned}
\vec{p_1} &= p_o\hat{\mathbf{z}}e^{-i\omega t} & \vec{a_1} &= \ell\hat{\mathbf{y}}\\
\vec{p_2} &= p_o\hat{\mathbf{z}}e^{-i\omega t} & \vec{a_2} &= \vec{a_1} - a\hat{\mathbf{y}}\\
\vec{p_3} &= p_o\hat{\mathbf{z}}e^{-i\omega t} & \vec{a_3} &= \vec{a_1} - 2a\hat{\mathbf{y}}\\
\ldots &= \ldots & & \\
\vec{p_N} &= p_o\hat{\mathbf{z}}e^{-i\omega t} & \vec{a_N} &= \vec{a_1} - (N-1)a\hat{\mathbf{y}}
\end{aligned}$$

We want to compute the intensity in the same geometry as discussed in the example 2.4 . The electric field is given by (35) and

$$\begin{aligned}
\vec{P}^{new}_{o\perp} =&\vec{p}_{o1,\perp} + e^{-i\vec{k}.(\vec{a}_2-\vec{a}_1)}\vec{p}_{o2,\perp} + \ldots + e^{-i\vec{k}.(\vec{a}_N-\vec{a}_1)}\vec{p}_{oN,\perp}\\
=&p_o\hat{\mathbf{z}}_\perp(1 + e^{iaksin\phi} + e^{i2aksin\phi} + \ldots + e^{i(N-1)aksin\phi})
\end{aligned} \tag{67}$$

The sum in parenthesis in Eq.(67) is a geometric series

$$S_{N-1} = \sum_{J=0}^{N-1} \eta^J = \frac{1-\eta^N}{1-\eta} \quad \text{where } \eta = e^{i\alpha} \text{ and } \alpha = aksin\,\phi, \tag{68}$$

and S_{N-1} can be written as

$$S_{N-1} = e^{i\frac{\alpha}{2}(N-1)} \left(\frac{sin\frac{N\alpha}{2}}{sin\frac{\alpha}{2}} \right) . \tag{69}$$

Thus the electric field in the x, y plane is (see also Eq.(35))

$$\vec{E} = \left(\frac{1}{4\pi\epsilon_o} \right) e^{-i\vec{k}.\vec{a}_1} \, p_o \, \hat{\mathbf{z}} \, e^{i\frac{\alpha}{2}(N-1)} \left(\frac{sin\frac{N\alpha}{2}}{sin\frac{\alpha}{2}} \right) \left\{ \frac{e^{i(kr-\omega t)}}{r} \right\} \tag{70}$$

where we used the fact that on the x, y plane $\hat{\mathbf{z}}_\perp = \hat{\mathbf{z}}$. The intensity can be computed as in (61)

$$I(\hat{\mathbf{r}}) =< \vec{S} > .\hat{\mathbf{r}} = \frac{1}{r^2} \frac{d\overline{P}}{d\Omega} = I_o \left(\frac{sin\frac{N\alpha}{2}}{sin\frac{\alpha}{2}} \right)^2 . \tag{71}$$

In order to simplify the analysis of this intensity formula, let us restrict ourselves to small angles so that

$$\frac{\alpha}{2} = \frac{ak}{2} sin\phi \approx \frac{ak}{2} tan\phi = \frac{\pi a}{\lambda} \frac{h}{L} = \pi \frac{h}{h_o} \quad \text{where } h_o = \tfrac{\lambda L}{a} . \tag{72}$$

Let us check this formula for the cases we know

1. $N = 1$. In that case $I(\hat{\mathbf{r}}) = I = o$.

2. $N = 2$. In that case we have

$$I(\hat{\mathbf{r}}) = I_o \left(\frac{sin\frac{2\alpha}{2}}{sin\frac{\alpha}{2}} \right)^2 = 4I_o cos^2 \frac{\alpha}{2} = 4I_o cos^2 \pi \frac{h}{h_o} \tag{73}$$

   which is our old formula (65). Notice that maxima occur at $h = nh_o$ with $n = 0, \pm 1, \pm 2, \ldots$ and minima at $h = (n + \frac{1}{2})h_o$.

We now consider the N-dipole array formula (71):

1. As $h \to 0$, that means $\frac{\alpha}{2} \to 0$, then $I = N^2 I_o$. This is because $sinx \to x$ for $x << 1$.

2. What happens to the formula at zeros of the denominator other than $h = 0$, that is for $sin\frac{\alpha}{2} \to 0$ for $\frac{\alpha}{2} \to n\pi$? Let us consider this as a

limiting procedure, and set $\frac{\alpha}{2} \to n\pi + \epsilon$ with $|\epsilon| << 1$. Since

$$\begin{aligned} sin\frac{\alpha}{2} &= sin\, n\pi \; cos\, \epsilon + cos\, n\pi \; sin\, \epsilon \approx \epsilon cos\, n\pi = (-)^n \epsilon \\ sin\, \frac{N\alpha}{2} &= sin\, Nn\pi \; cos\, N\epsilon + cos\, Nn\pi \; sin\, N\epsilon \\ &\approx N\epsilon\, cos\, Nn\pi = (-)^{Nn} N\epsilon \end{aligned} \tag{74}$$

Inserting these limiting values in Eq.(71) we obtain

$$I = I_o \left(\frac{(-)^{Nn} N\epsilon}{(-)^n \epsilon} \right)^2 = N^2 I_o \,. \tag{75}$$

Thus we conclude that as $\frac{\alpha}{2} \to n\pi$, which is the same thing as $h \to nh_o$ (where $h_o = \lambda a/L$), intensity reaches maximum value $I \to N^2 I_o$. They are called *principal maxima*. The positions of these maxima are the same as for the case $N = 2$, but with the following differences

i) Intensity fringes are much brighter (instead of $4I_o$ we now have $N^2 I_o$.

ii) Interference fringes are much sharper as we discuss below:

Between two principal maxima there are zeros of intensity when $sin\, \frac{N\alpha}{2} = 0$.

$$\frac{N\alpha}{2} = N\frac{\pi h}{h_o} = m\pi \,, \quad m = 1, 2, \ldots, N-1. \tag{76}$$

Note that in Eq.(76) the values $m = 0$ or $m = N$ are missing since they correspond to the positions of principal maxima. Thus in betwen neighboring two principal maxima the distance is h_o and, there are $N - 1$ intensity zeros at the positions $\frac{h_o}{N}, 2\frac{h_o}{N} \ldots, (N-1)\frac{h_o}{N}$.

Obviously, in between $N-1$ zeros of intensity, there must be $N-2$ maxima. What are their positions and what is the intensity at such points? Again to simplify the computation, if we consider N large, so that in between adjacent zeros denominator $sin\, \frac{\alpha}{2}$ can be treated as a constant, then the positions of the maxima are at $sin\, \frac{\alpha}{2} = sin\, \frac{\pi h}{h_o} = \pm 1$, which gives $h = (n + \frac{1}{2}) \left(\frac{h_o}{N}\right)$ where $n = 1, 2, \ldots, N-2$, as secondary maxima must fall in between the zeros of intensity. This calculation is only very approximate and we can guess that the secondary maxima will not fall exactly in the middle of two zeros. As for the magnitude of the secondary maxima, here is an exercise:

Exercise 4.

1. Show that the intensity of the *adjacent* secondary maxima are, for large N, can be approximated as,

$$\begin{aligned}\frac{I_{sm}(n=1)}{N^2 I_o} &= \frac{I_{sm}(n=N-2)}{N^2 I_o} \approx \frac{1}{22}\,, \\ \frac{I_{sm}(n=2)}{N^2 I_o} &= \frac{I_{sm}(n=N-3)}{N^2 I_o} \approx \frac{1}{62}\,.\end{aligned} \tag{77}$$

2. Give a *qualitative* argument for the following questions

   (a) Why do the secondary maxima adjacent to the principal maxima have higher intensities than the secondary maxima that fall in the region midway between principal maxima?

   (b) The secondary maxima have their peaks not eactly in the middle of two adjacent zeros, but closer to the side of the nearest primary maximum, why?

   In the following we will ignore these fine points, take the maxima at the middle of the two adjacent intensity zeros, and also treat intensities there as being just quite small compared to the principal maxima.

3. What is the width of a principal maximum? (Ans: Check the two zeros at both sides of the principal maximum and find the width as $\Delta = 2\,\frac{h_o}{N}$).

End of Exercise 4

With all these simplifying conditions, our interference pattern repeats itself at every h_o. At $h = nh_o$ we have principal maxima where intensity is $I = N^2 I_o$. In between there are $N-1$ zeros at positions $\frac{h_o}{N}, 2\frac{h_o}{N}, \ldots (N-1)\frac{h_o}{N}$. In between these zeros there are secondary maxima, with positions approximately at $\frac{3}{2}\frac{h_o}{N}, \frac{5}{2}\frac{h_o}{N}, \ldots (N-\frac{3}{2})\frac{h_o}{N}$ with intensities much smaller than the principal maxima. And the width of the principal maxima is $\Delta = 2\,\frac{h_o}{N}$.

In the Figure (13) below you see a periodic cell in the interference pattern. Clearly, N-dipole result simulates the N-slit interference experiment in optics.

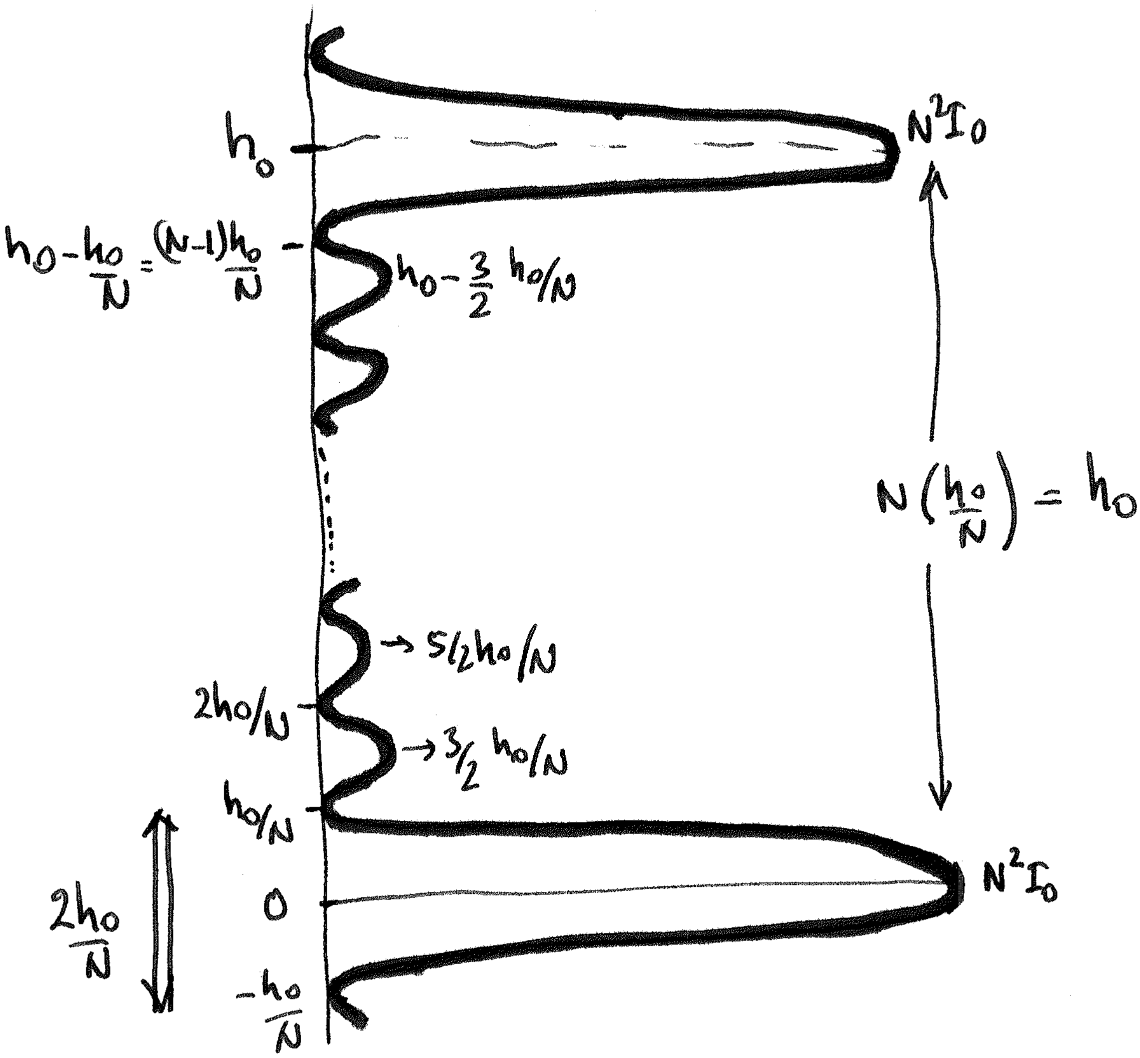
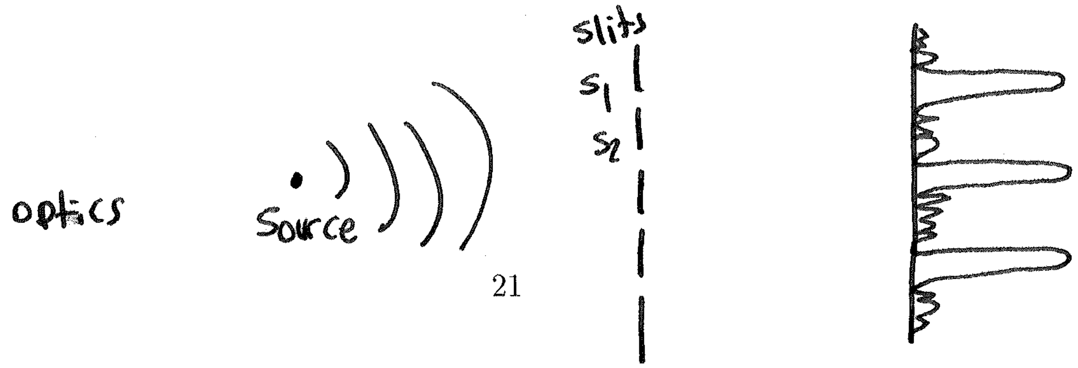


Figure 13: Interference pattern due to N in-phase dipoles



3. DIFFRACTION

3.1 Single slit diffraction

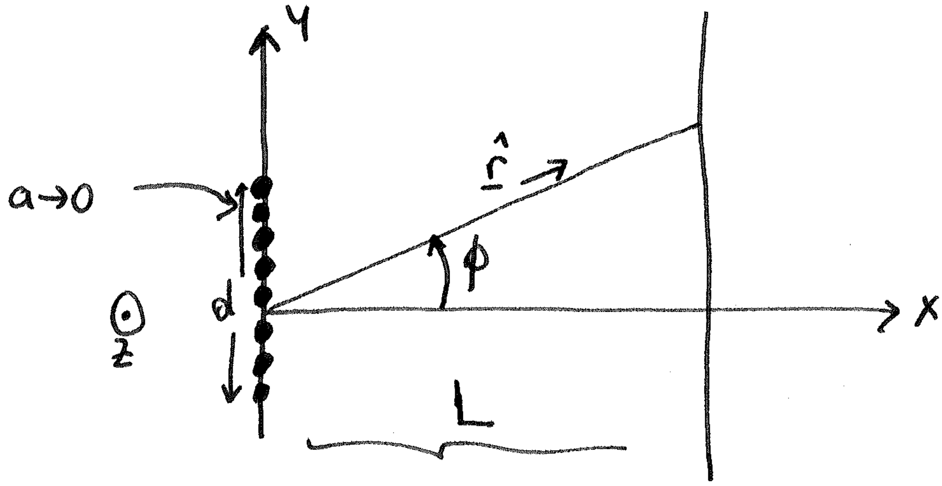


Figure 14: Continuous in-phase dipole distribution parallel to the z axis

We consider a continuous distribution of dipoles such that

$$\begin{aligned} &N \to \infty\,, \quad p_o \to 0\,, \quad \text{such that } Np_o \to P_o \text{ is finite,} \\ &N \to \infty\,, \quad a \to 0\,, \quad \text{such that } Na \to d \text{ is finite.} \end{aligned} \tag{78}$$

We consider the electric field of the N-in phase dipoles given in (70) and we see that it is convenient to make the following definitions

$$\begin{aligned} \frac{N\alpha}{2} = \frac{Naksin\,\phi}{2} \to \frac{\beta}{2} = \frac{kdsin\,\phi}{2} \approx \frac{kdtan\,\phi}{2} = \frac{kd}{2}\frac{H}{L} \\ \frac{\beta}{2} = \pi\,\frac{H}{H_o} \quad \text{where } H_o = \tfrac{\lambda L}{d} \\ sin\,\frac{\alpha}{2} = sin\,\frac{\beta}{2N} \approx \frac{\beta}{2N} \end{aligned} \tag{79}$$

We write the electric field with these new definitions

$$\begin{aligned} \vec{E} &= \left(\frac{1}{4\pi\epsilon_o}\right) e^{-i\vec{k}.\vec{a}_1}\, p_o\,\hat{\mathbf{z}}\; e^{i\frac{\alpha}{2}(N-1)} \left(\frac{sin\frac{N\alpha}{2}}{sin\frac{\alpha}{2}}\right) \left\{\frac{e^{i(kr-\omega t)}}{r}\right\} \\ &= \left(\frac{1}{4\pi\epsilon_o}\right) e^{-i\vec{k}.\vec{a}_1}\, P_o\,\hat{\mathbf{z}}e^{i\frac{\beta}{2}} \underbrace{e^{-i\frac{\beta}{2N}}}_{+1} \left(\frac{sin\frac{\beta}{2}}{\frac{\beta}{2}}\right) \left\{\frac{e^{i(kr-\omega t)}}{r}\right\} \end{aligned} \tag{80}$$

Now the intensity can be computed easily

$$I(\hat{\mathbf{r}}) = < \vec{S} > .\hat{\mathbf{r}} = \frac{1}{r^2}\frac{d\overline{P}}{d\Omega} = I_o \left(\frac{sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^2 . \tag{81}$$

with I_o representing the constant intensity due to a single dipole of strength $|P_o|$ at the origin,

$$I_o = \left(\frac{1}{4\pi\epsilon_o}\right)\frac{ck^4|P_o|^2}{8\pi L^2} \tag{82}$$

We now study the angular distribution of intensity given by Eq.(81):

1. When $\beta \to 0$, $I \to I_o$, the principal maximum.

2. Diffraction minima occurs when the intensity vanishes, $I = 0$. This will happen when $sin\frac{\beta}{2} = 0$, or in other words when

$$\frac{\beta}{2} = n\pi \quad \text{or} \quad \frac{\beta}{2} = \pi\frac{H}{H_o} = n\pi \tag{83}$$

   Thus for $H = nH_o$ with $n = \pm 1, \pm 2\ldots$ we have zero intensity. (Note that H=0 is the position of the principal maximum, and thus $n = 0$ is to be excluded.)
   The width of the principal maximum is $2H_o$, and secondary maxima are of width H_o.

3. The secondary maxima correspond to the maximum values of I beyond $\pm H_o$. Again, to simplify the computation, we assume they occur near the maximum values of the sin function $sin\,\frac{\beta}{2} = \pm 1$, this gives

$$\begin{aligned} &\frac{\beta}{2} = \pm\,(2n+1)\frac{\pi}{2} \quad \text{where } n = 1, 2, .., \text{ and} \\ &H = ...\frac{5}{2}H_o, \frac{3}{2}H_o, -\frac{3}{2}H_o, -\frac{5}{2}H_o... \end{aligned} \tag{84}$$

   Actually because of the term $\left(\frac{\beta}{2}\right)^2$, the positions above do not exactly correspond to the positions of secondary maxima. The positions of the secondary maxima given above are therefore only approximate.

In the Figure (15) below you see the diffraction pattern due to a set of continuous dipole distribution. Clearly, this dipole distribution result simulates the single-slit diffraction pattern in optics.

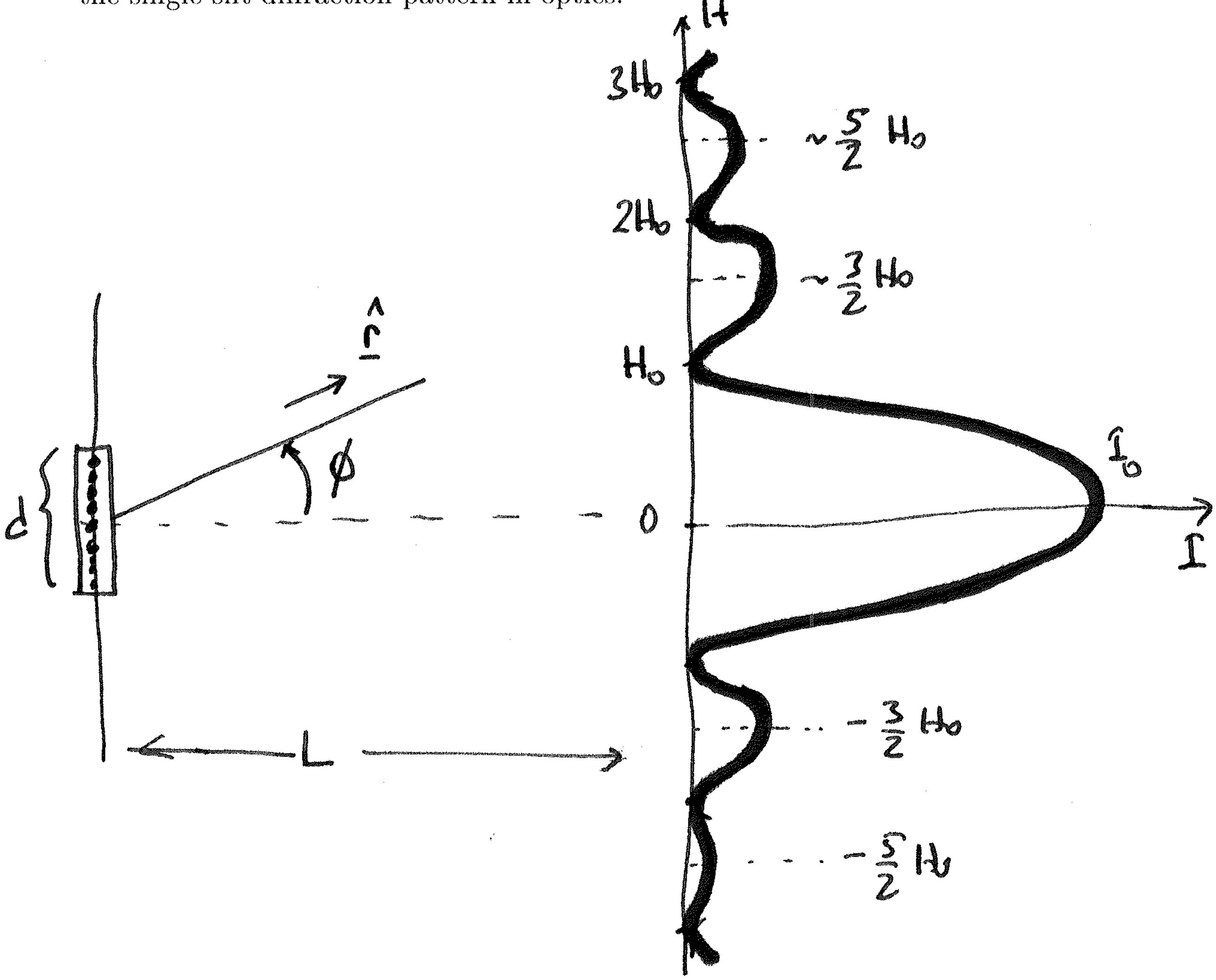
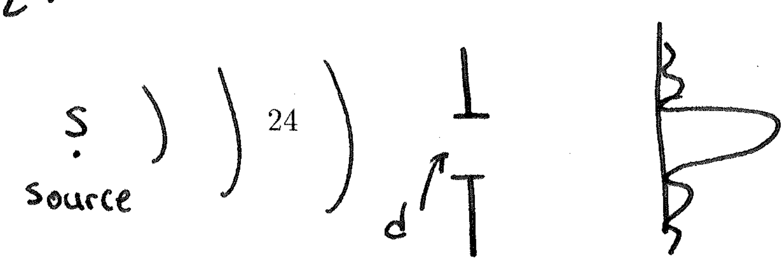


Figure 15: Diffraction pattern due to a continuous distribution of dipoles

$$I = I_0 \left(\frac{\sin \beta/2}{\beta/2} \right)^2 , \quad \frac{\beta}{2} = \pi \frac{H}{H_0} \qquad H_0 = \frac{\lambda L}{d}$$

optics :



An interesting point: Note that in the diffraction pattern as $d \to 0$ we have the positions of the minima $\pm H_o = \pm\frac{\lambda L}{d} \to \pm$large. As you make the hole smaller, and try forcing the light not to spread, exactly the opposite happens. Instead of going straight through the small hole, light bends more.

Quantal explanation:

1. Quantum theory of light [3] tells us that by making the hole bigger more paths are allowed. Each path is assocaited with a complex number. Interference among complex numbers corresponding to the paths not assocaited with straight lines cancel. Otherwise, when the hole is smaller, there are not enough paths, and cancellation is not complete, and light bends.

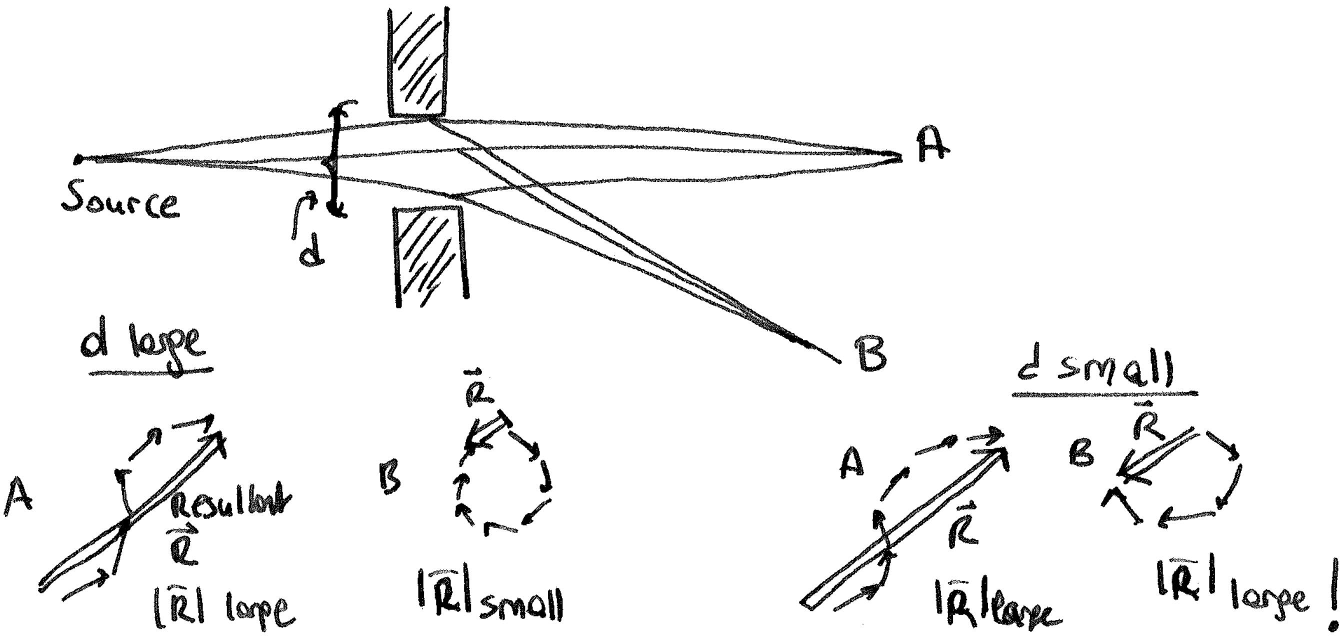


Figure 16: consider interference and cancellation of amplitudes representing paths of light [3]

2. You will also study this spreading of light when the hole gets smaller via the *uncertainty principle* obeyed by the photons later in the course.

3.2 Multiple slit diffraction

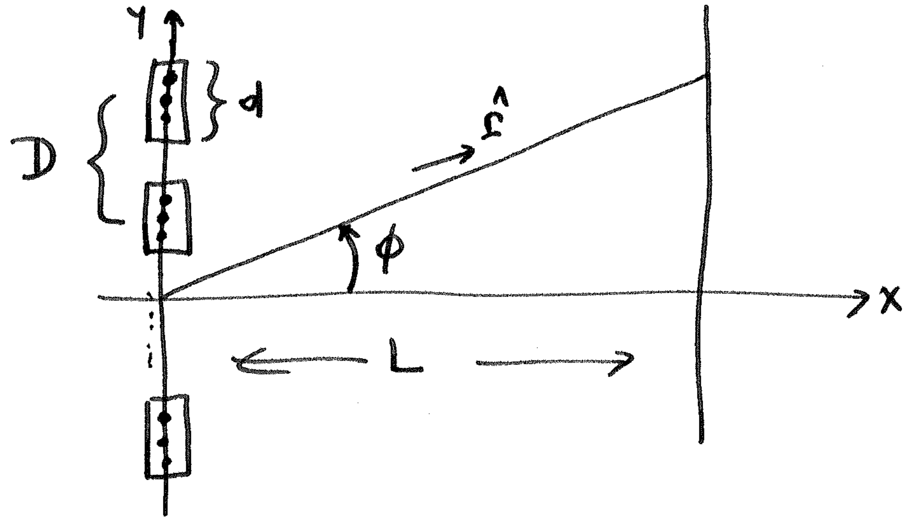


Figure 17: Group of continuous in-phase dipole distributions parallel to the z axis

We now consider N groups of continuous set of dipoles (N is a finite number now). The electric field due to a single group whose first dipole is at $\vec{a}_1$ was computed in 3.1 and given by Eq.(80). We add the electric fields of the rest of the groups using Eq.(35)

$$\begin{aligned}\vec{E}_{Total} = & \left(\frac{1}{4\pi\epsilon_o}\right) k^2 \, e^{i\frac{\beta}{2}} \left(\frac{sin\frac{\beta}{2}}{\frac{\beta}{2}}\right) \times \\ & e^{-i\vec{k}.\vec{a}_1} \, P_o\hat{\mathbf{z}} \left(1 + e^{-i\vec{k}.(\vec{a}_2-\vec{a}_1)} + \ldots + e^{-i\vec{k}.(\vec{a}_N-\vec{a}_1)}\right) \times \\ & \left\{\frac{e^{i(kr-\omega t)}}{r}\right\}\end{aligned} \tag{85}$$

The middle line above represents the contribution of different groups, and was evaluated in Eq.(70). Thus

$$\begin{aligned}\vec{E}_{Total} = & \left(\frac{1}{4\pi\epsilon_o}\right) k^2 \, e^{i\frac{\beta}{2}} \left(\frac{sin\frac{\beta}{2}}{\frac{\beta}{2}}\right) \times \\ & e^{-i\vec{k}.\vec{a}_1} \, P_o\hat{\mathbf{z}} \, e^{i\frac{\alpha}{2}(N-1)} \left(\frac{sin\frac{N\alpha}{2}}{sin\frac{\alpha}{2}}\right) \times \\ & \left\{\frac{e^{i(kr-\omega t)}}{r}\right\}\end{aligned} \tag{86}$$

The intensity is easy to compute now

$$I(\phi) = \underbrace{\left[I_o \left(\frac{sin\frac{\beta}{2}}{\frac{\beta}{2}} \right)^2 \right]}_{\text{single slit diffraction}} \quad \underbrace{\left(\frac{sin\frac{N\alpha}{2}}{sin\frac{\alpha}{2}} \right)^2}_{N \text{ slit interference}} \tag{87}$$

$$\text{where} \quad \frac{\beta}{2} = \frac{\pi d}{\lambda} sin\,\phi \approx \pi \frac{h}{h_o} \quad \text{and} \quad \frac{\alpha}{2} = \frac{\pi D}{\lambda} sin\,\phi \approx \pi \frac{H}{H_o}$$

$$H_o = \frac{\lambda L}{d} \qquad h_o = \frac{\lambda L}{D}$$

and d is the length of a continuous dipole distribution, and D is the distance between distributions.

Clearly this simulates the N-slit interference in the presence of diffraction, in optics.

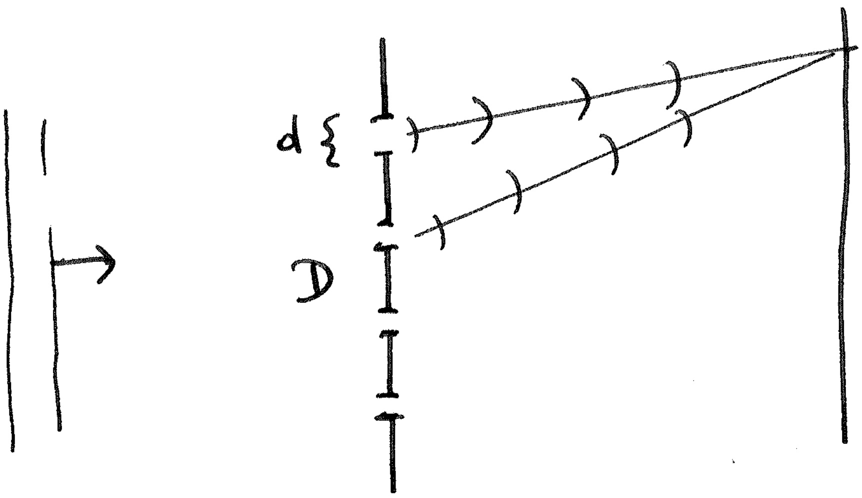
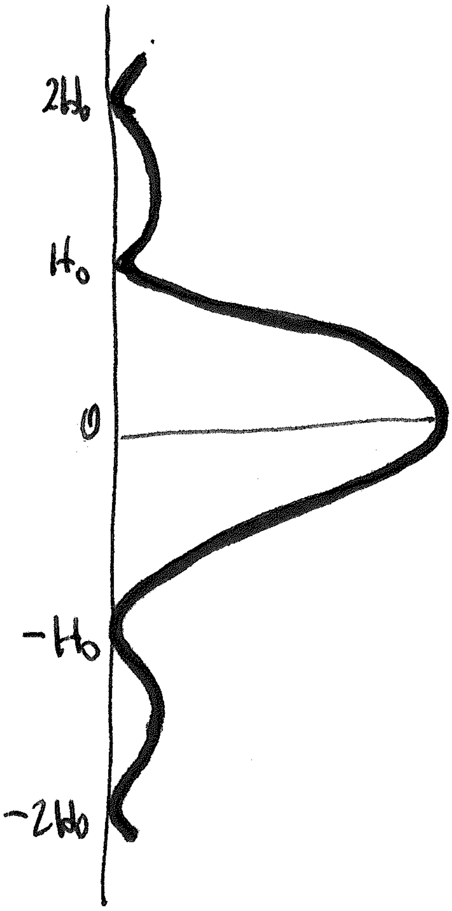


Figure 18: N slit interference with diffraction

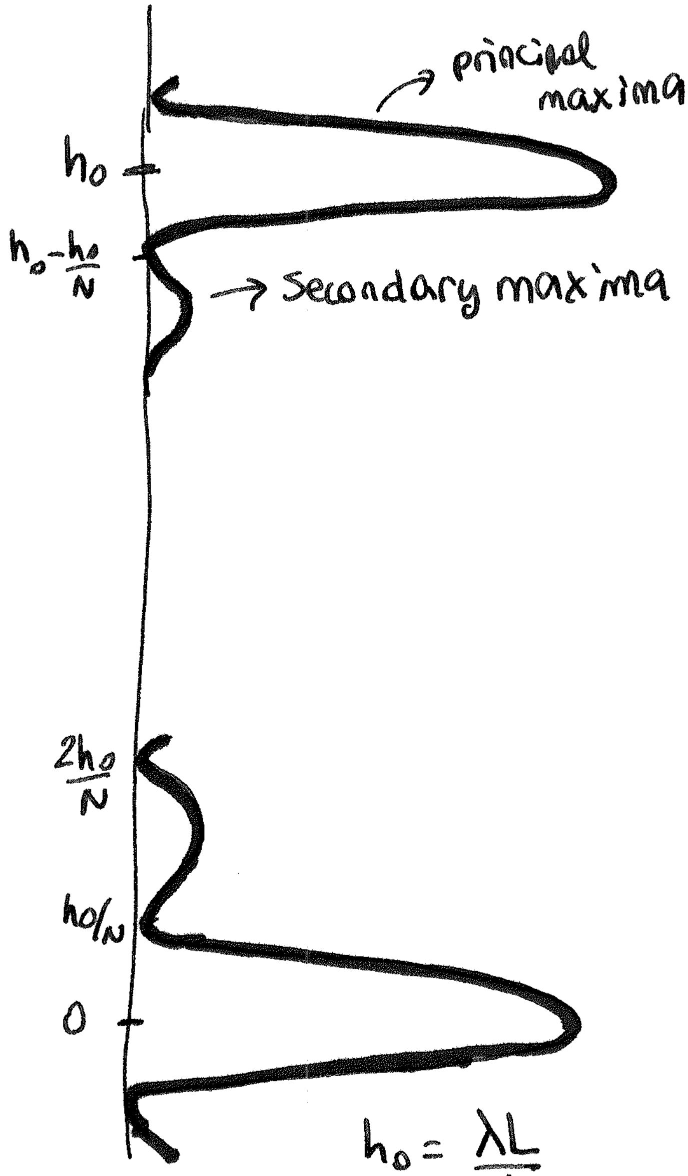
Eq.(87) means that diffraction provides an envelope on the N slit interference pattern. In the following you will see seperate sketches for the envelope intensity and interference pattern , and a sketch of interference pattern under the diffraction envelope.



$H_0 = \frac{\lambda L}{d}$

d = slit width

diffraction



$h_0 = \frac{\lambda L}{D}$

D = distance between slits

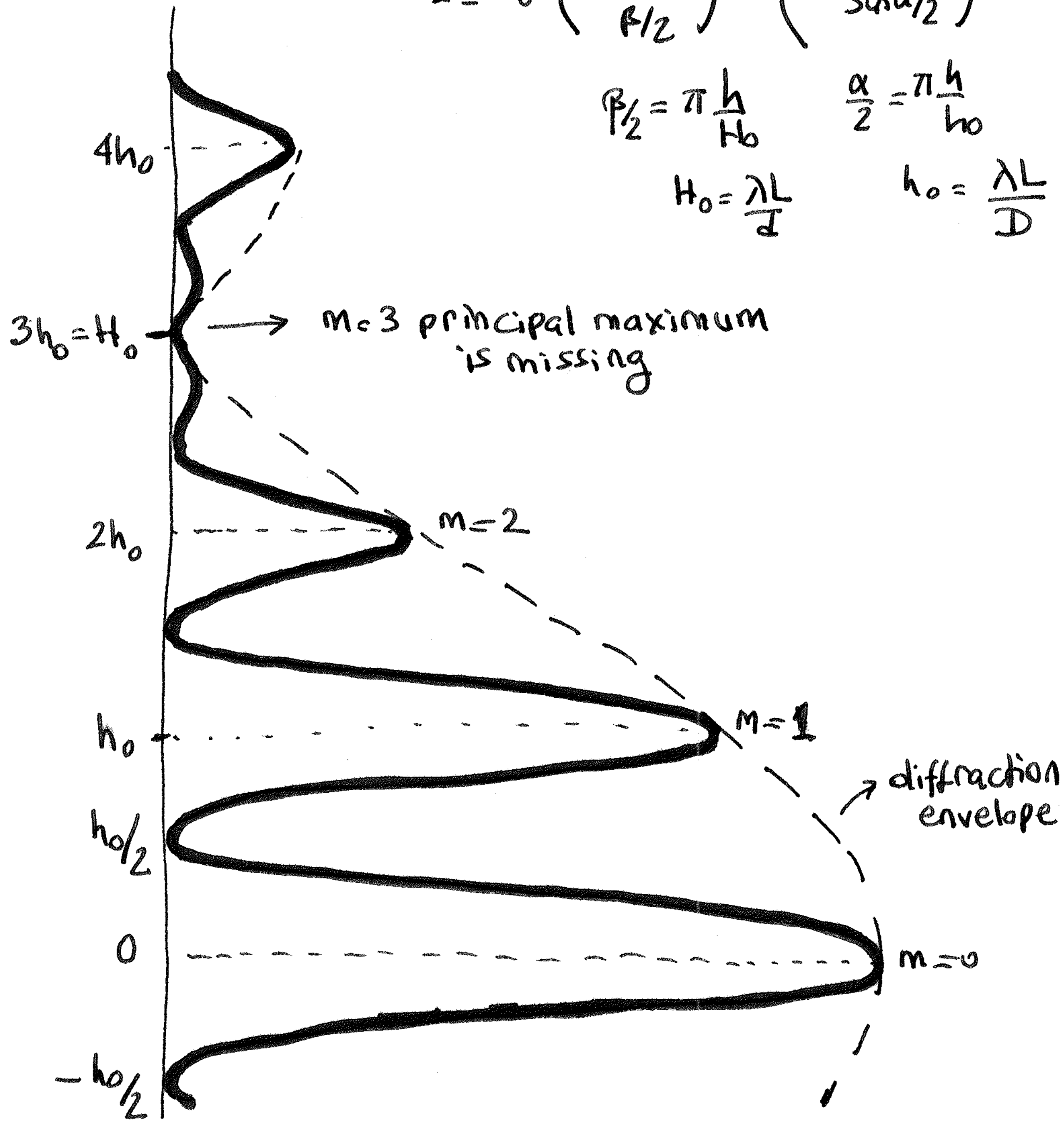
interference

$N = 2 \qquad H_0 = 3h_0 \quad (\text{i.e., } D = 3d)$

$$I = I_0 \left(\frac{\sin \beta/2}{\beta/2} \right)^2 \left(\frac{\sin N\alpha/2}{\sin \alpha/2} \right)^2$$

$$\beta/2 = \pi \frac{h}{H_0} \qquad \frac{\alpha}{2} = \pi \frac{h}{h_0}$$

$$H_0 = \frac{\lambda L}{d} \qquad h_0 = \frac{\lambda L}{D}$$



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[2] George Bekefi and Alan H. Barett. *Electromagnetic Vibrations, Waves and Radiation*. MIT Press., 1990.

[3] Richard P. Feynman. *QED: The Strange Theory of Light and Matter*. Princeton University Press., 1988.