Phys. 401
Fall 2009

Liénard-Wiechert Potentials
and
Fields for a point charge

1) Start from

$$\phi(\bar{r},t) = \frac{1}{4\pi\epsilon_0} \int d\bar{r}' dt' \, \rho(\bar{r}',t') \frac{\delta\left(t' - \left[t - \frac{|\bar{r}-\bar{r}'|}{c}\right]\right)}{|\bar{r}-\bar{r}'|}$$

and $$\vec{A}(\bar{r},t) = \frac{\mu_0}{4\pi} \int d\bar{r}' dt' \, \vec{j}(\bar{r}',t') \frac{\delta\left(t' - \left[t - \frac{|\bar{r}-\bar{r}'|}{c}\right]\right)}{|\bar{r}-\bar{r}'|}$$

and for a single particle $\rho(\bar{r},t) = q\,\delta(\bar{r}-\bar{r}_0(t))$

$$\vec{j}(\bar{r},t) = \rho(\bar{r},t)\,\bar{v}(t)$$

$$\bar{v}(t) = \frac{d\bar{r}_0}{dt}$$

and obtain

$$\boxed{\begin{aligned} \phi(\bar{r},t) &= \frac{q}{4\pi\epsilon_0} \frac{1}{R - \vec{R}\cdot\vec{\beta}} \\ \vec{A}(\bar{r},t) &= \frac{1}{c} \frac{q}{4\pi\epsilon_0} \frac{\vec{\beta}}{R - \vec{\beta}\cdot\vec{R}} \end{aligned}}$$

LW potentials

$$\vec{R} = \bar{r} - \bar{r}_0(t_R) = \bar{R}(t_R)$$

$$R = |\vec{R}| = R(t_R)$$

$$t_R = t - \frac{R(t_R)}{c}$$

LW_2

2) Compute:

i) $$\frac{\partial t_R}{\partial t} = \frac{1}{1-\underline{\hat{n}}\cdot\vec{\beta}},$$

$$\underline{\hat{n}} = \frac{\vec{R}}{R} = \underline{\hat{n}}(t_R)$$

$$\vec{\beta} = \frac{\vec{v}(t_R)}{c}$$

ii) $$\frac{1}{c}\frac{\partial R}{\partial t} = -\frac{\underline{\hat{n}}\cdot\vec{\beta}}{1-\hat{n}\cdot\vec{\beta}}$$

iii) $$\nabla t_R = \frac{-\hat{n}/c}{1-\hat{n}\cdot\vec{\beta}}$$

iv) $$\nabla R = \frac{\hat{n}}{1-\hat{n}\cdot\vec{\beta}}$$

3) $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$

a) $-\nabla\phi = -\nabla\phi\Big|_{t_R = \text{const}} - \frac{\partial t_R}{\partial \vec{r}} \frac{\partial \phi}{\partial t_R}$

$$-\nabla\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{R^2} \frac{1}{(1-\hat{n}.\vec{\beta})^3} \left(\hat{n}(1-\beta^2) - \vec{\beta}\,(1-\hat{n}.\vec{\beta})\right)$$

$$+ \frac{q}{4\pi\epsilon_0 c^2} \frac{1}{R} \frac{1}{(1-\hat{n}.\vec{\beta})^3} \hat{n}\ \hat{n}.\vec{a}$$

b) $$-\frac{\partial \vec{A}}{\partial t} = \frac{q}{4\pi\epsilon_0 c^2} \frac{1}{(1-\hat{n}.\vec{\beta})^3} \left\{ \frac{-\vec{a} + \hat{n}\times(\vec{a}\times\vec{\beta})}{R} + \frac{\vec{v}\, v^2/c - \vec{v}\ \hat{n}.\vec{v}}{R^2} \right\}$$

Combine (a) and (b), $\vec{E} = \vec{E}_{\text{acceleration}} + \vec{E}_{\text{velocity}}$

$$\vec{E} = \frac{q}{4\pi\epsilon_0 c^2} \frac{1}{R} \frac{1}{(1-\hat{n}.\vec{\beta})^3} \left(-\vec{a}_\perp + \hat{n}\times(\vec{a}\times\vec{\beta})\right)$$

$$+ \frac{q}{4\pi\epsilon_0} \frac{1}{R^2} \frac{1}{(1-\hat{n}.\vec{\beta})^3} \left((\hat{n}-\vec{\beta})(1-\beta^2)\right)$$

LW_4

4) $$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{A}\Big|_{t_R=\text{const}} + \nabla t_R \times \frac{\partial \vec{A}}{\partial t_R}$$

i) $$\nabla \times \vec{A}\Big|_{t_R} = -\frac{q}{4\pi\epsilon_0}\frac{1}{c}\frac{(\hat{n}-\vec{\beta})\times\vec{\beta}}{R^2(1-\hat{n}\cdot\vec{\beta})^2}$$

ii) $$\nabla t_R \times \frac{\partial \vec{A}}{\partial t_R} = \frac{q}{4\pi\epsilon_0 c^2}\frac{(-1)\ \hat{n}/c}{(1-\vec{\beta}\cdot\hat{n})^2} \times \left\{ \vec{a}/R + \frac{\vec{v}\ \hat{n}\cdot\vec{v} - \vec{v}\ \vec{\beta}\cdot\vec{v}}{R^2(1-\vec{\beta}\cdot\hat{n})} + \frac{\vec{v}\ \vec{a}\cdot\hat{n}/c}{R(1-\vec{\beta}\cdot\hat{n})} \right\}$$

iii) $\vec{B} = \vec{B}_{acc} + \vec{B}_{vel}$

$$\vec{B}_{acc} = \frac{\hat{n} \times \vec{E}_{acc}}{c} \qquad \vec{B}_{vel} = \frac{\hat{n}\times\vec{E}_{vel}}{c}$$

so that $\boxed{c\vec{B} = \hat{n} \times \vec{E}}$

5) Show that $\hat{n}\cdot\vec{B}_{acc} = 0 \quad \hat{n}\cdot\vec{E}_{acc} = 0 \quad \vec{E}_{acc}\cdot\vec{B}_{acc} = 0$

$$\vec{E}_{acc} \sim \frac{1}{R} \qquad \vec{B}_{acc} \sim \frac{1}{R}$$

6) Feynman expression for the radiation formula:

We obtained in class:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \cdot \left\{ \frac{\hat{n}}{(1-\hat{n}\cdot\vec{\beta})R^2} + \frac{1}{1-\hat{n}\cdot\vec{\beta}} \frac{d}{c\,dt'} \left(\frac{\vec{R} - R\vec{\beta}}{R^2(1-\hat{n}\cdot\vec{\beta})} \right) \right\}_{t'=t_{ret}}$$

To arrive at Feynman formula we should convert

$$\frac{d}{dt'} \longrightarrow \frac{d}{dt}.$$

Note that $\displaystyle \frac{dt'}{dt} = \frac{d}{dt}\left(t - \frac{R}{c}\right) = 1 - \frac{\dot{R}}{c} = \frac{1}{1-\hat{n}\cdot\vec{\beta}}$

$\Uparrow$ we found this earlier. (eq. LW-2 – 2i)

$$\frac{\vec{E}}{q/4\pi\epsilon_0} = \frac{\hat{n}}{R^2}\left(1 - \frac{\dot{R}}{c}\right) + \underbrace{\frac{dt'}{dt}\frac{d}{c\,dt'}}_{\frac{d}{c\,dt}} \left(1 - \frac{\dot{R}}{c}\right)\left(\frac{\hat{n}}{R} - \frac{\vec{\beta}}{R}\right)$$

$$\frac{d}{c\,dt}\left[\frac{\hat{n}}{R} - \frac{\hat{n}\dot{R}}{cR} - \frac{(1-\dot{R}/c)}{R}\frac{(-)}{c}\frac{d\vec{R}}{dt'}\right]$$

where $\displaystyle \vec{\beta} = \frac{d\vec{r}_0}{c\,dt'} = -\frac{1}{c}\frac{d}{dt'}(\vec{r} - \vec{r}_0(t')) = -\frac{d\vec{R}}{dt'}$

$$= \frac{\hat{n}}{R^2} - \frac{\hat{n}\dot{R}}{R^2 c} + \frac{1}{c}\frac{d}{dt}\frac{\hat{n}}{R} - \frac{1}{c}\frac{d}{dt}\frac{\hat{n}\dot{R}}{cR}$$

$$+ \frac{d}{c\,dt}\frac{1}{R}\frac{d}{c\,dt}\vec{R} \quad , \qquad \vec{R} = R\hat{n}$$

a) Combine the <u>second</u> and <u>third terms</u>

$$-\frac{\hat{n}\dot{R}}{R^2 c} + \frac{1}{c}\frac{d}{dt}\frac{\hat{n}}{R} = \frac{R}{c}\frac{d}{dt}\frac{\hat{n}}{R^2}$$

b) Manipulate the remaining terms and find:

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left\{\frac{\hat{n}}{R^2} + \frac{R}{c}\frac{d}{dt}\left(\frac{\hat{n}}{R^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\hat{n}\right\}.$$

<u>Feynman's formula</u>.

7) Compute $\frac{d^2\hat{n}}{dt^2}$ and show that <u>acceleration</u> terms are the LW fields:

$$\frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\frac{d^2\hat{n}}{dt^2}\Bigg| = \frac{q}{4\pi\epsilon_0 c^2}\frac{1}{(1-\hat{n}\cdot\vec{\beta})^3}\left\{\frac{-\vec{a}_\perp + \hat{n}\times(\vec{a}\times\vec{\beta})}{R}\right\}$$

keep only the Acceleration terms