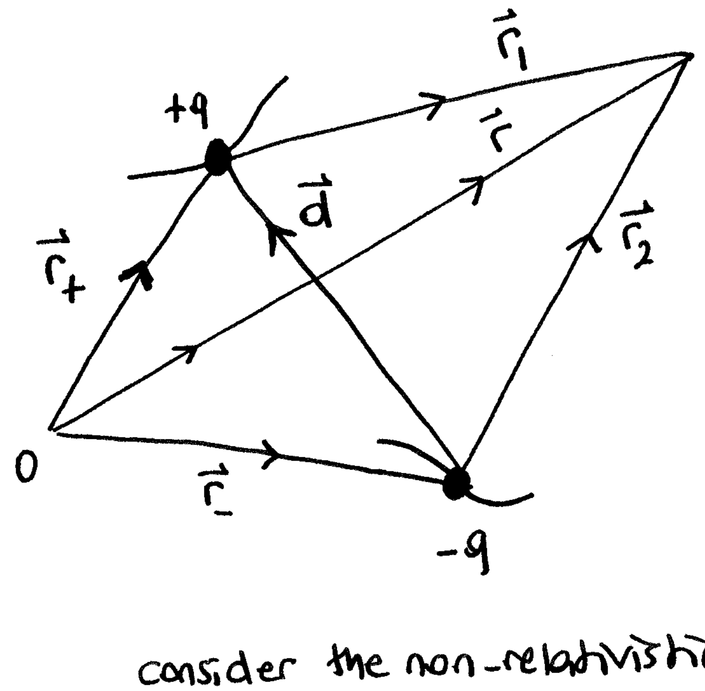
Phys. 402
Spring 2010
Home work 3

1) Point Electric Dipole (Non-relativistic)

If the charge $+q$ $(q>0)$ follows the trajectory $\vec{r}_+(t)$, and
" " " $-q$ " " " $\vec{r}_-(t)$

define $\vec{p} \equiv q(\vec{r}_+(t) - \vec{r}_-(t)) = q\vec{d}(t)$

such that as $q \to \infty$ and $|\vec{d}| = d \to 0$ such that $|\vec{p}| = p$ is finite $\Longrightarrow$ point electric dipole definition.



(A) Superpose the LW-electric fields given in the LW-Homework for $+q$ and $-q$:
$\vec{E} = \vec{E}_+ + \vec{E}_-$

consider the non-relativistic approximation

$$|\vec{\beta}_+| = \left|\frac{\vec{v}_+}{c}\right| \ll 1 \quad |\vec{\beta}_-| = \left|\frac{\vec{v}_-}{c}\right| \ll 1 \text{, and,}$$

obtain the electric field of a (non-relativistic) point electric dipole:

$$\boxed{\vec{E}(\vec{r},t)_{\text{point electric dipole}} = \frac{1}{4\pi\epsilon_0}\left[\frac{3\vec{p}^{*}\cdot\hat{\underline{R}}\,\hat{\underline{R}} - \vec{p}^{*}}{R^3}\right] - \frac{1}{4\pi\epsilon_0 c^2}\frac{\ddot{\vec{p}}_{\perp}(t-R/c)}{R}}$$

where $\vec{p}$ = point electric dipole

$\vec{R} = \vec{r} - \underline{\vec{r}}(t_R)$, $\hat{R} = \vec{R}/|\vec{R}|$ $R = |\vec{R}|$

and $\vec{p}^{*} = \vec{p}\,(t - R/c) + \frac{R}{c}\,\dot{\vec{p}}\,(t-R/c)$

where $\vec{p}^{*}$ is the dipole moment <u>retarded</u> ($\vec{p}(t-R/c)$ term) then <u>corrected for the retardation</u> ($\frac{R}{c}\dot{\vec{p}}(t-R/c)$ term).

(B) Show that

$$\boxed{\vec{B}(\vec{r},t)_{\text{point electric dipole}} = \frac{1}{4\pi\epsilon_0}\frac{1}{c^2}\frac{\dot{\vec{p}}^{*}\times\hat{\underline{R}}}{R^2}}$$ The magnetic field of a non-relativistic dipole.

where $\dot{\vec{p}}^{*} = \left(\dot{\vec{p}} + \frac{R}{c}\ddot{\vec{p}}\right)_{t-R/c} = \dot{\vec{p}}(t-R/c) + \frac{R}{c}\ddot{\vec{p}}\,(t-R/c)$

Comments and hints

i) Notice that for $\vec{E}$ field close to the dipole, the term in the brackets is dominant. The first order effects of the delay are cancelled by the second term in $\vec{p}^*$. Hence in the near zone (where the instantaneous dipole moment $\vec{p}(t)$ is used) the static formulas are quite accurate.

ii) If $|\vec{r}| \gg |\vec{r}_-|$, then R may be replaced by r.

iii) Hint: The term in the brackets in $\vec{E}$ arises from the $\vec{E}_{\vec{v}_+} + \vec{E}_{\vec{v}_-}$ parts of the LW-E fields. When you add the $\vec{E}_v$ fields for $(+q)$ and $(-q)$ the lowest order terms that survive in the non-relativistic approximation are those which contain $\frac{d}{R} \ll 1$ and $\frac{v}{c} \ll 1$ terms. (that is, linearize $E_{\vec{v}_+} + E_{\vec{v}_-}$ in β and d/R).

iv) Hint: $c\vec{B}_{\substack{\text{point}\\\text{electric dipole}}} \neq \hat{R} \times \vec{E}_{\substack{\text{point}\\\text{dipole}}}$. explain.

What is the correct way to compute $\vec{B}_{\substack{\text{point}\\\text{electric}\\\text{dipole}}}$?

2) Point Magnetic Dipole (non-relativistic)

Show that as $\vec{p} \rightarrow \vec{m}/c$

$$\vec{E}^{E1}(\vec{p}) \longrightarrow c\,\vec{B}^{M1}(\vec{m}/c)$$

$$\vec{B}^{E1}(\vec{p}) \longrightarrow -\frac{\vec{E}^{M1}}{c}(\vec{m}/c)$$

gives :

$$\boxed{\begin{aligned} \underset{\text{point magnetic dipole}}{\vec{B}(\vec{r},t)} &= \frac{\mu_0}{4\pi}\frac{3\,\vec{m}^*\cdot\hat{R}\,\hat{R}-\vec{m}^*}{R^3} - \frac{\mu_0}{4\pi c^2}\frac{\ddot{\vec{m}}_\perp(t-R/c)}{R} \\ \underset{\text{point magnetic dipole}}{\vec{E}(\vec{r},t)} &= \frac{\mu_0}{4\pi}\frac{\hat{R}\times\dot{\vec{m}}^*}{R^2} \end{aligned}}$$

where $\vec{m}^* = \vec{m}(t-R/c) + \frac{R}{c}\dot{\vec{m}}(t-R/c)$

3) Electric dipole radiation by an oscillating point charge

a) Show that sources $\rho(\vec{r},t)$ and $\vec{j}(\vec{r},t)$ with definite frequency ω will produce fields of that frequency (i.e., if $\rho(\vec{r},t) = \rho_0(\vec{r})\, e^{-i\omega t}$ etc.) (Hint: Use Maxwell eqs).

b) A point charge q oscillates in the z-direction with displacement from the origin given as

$$\vec{r}_0(t) = D \cos\omega t\, \hat{z} \, .$$

Show that an oscillating point charge possesses not only the frequencies of its motion, but all its overtones (i.e., $\omega, 2\omega, 3\omega \ldots$).

(Hint: Plot $\rho(z_0,t)$ as a function of time for $|z| < D$, $z_0 =$ constant. Discuss that $\rho(z_0,t)$ is periodic, but not harmonic.

c) Since $\rho(\vec{r},t) = q\,\delta(\vec{r} - \vec{r}_0(t))$, find the current density $\vec{j}(\vec{r},t)$. Show that charge is conserved, i.e., the equation of continuity is satisfied.

d) For the oscillations with small amplitude, the charge density can be approximated by

$$\rho(\vec{r},t) \cong q\,\delta(\vec{r}) - q\,(\vec{r}_0(t)\cdot\vec{\nabla})\,\delta(\vec{r})$$

What is the charge conserving current density for this approximation? (You must make sure that the approximate forms of ρ and $\vec{j}$ satisfy the equation continuity).

e) With the approximate source terms of part (d) above, find the vector and scalar potentials of the Lorentz gauge:

$$\vec{A}(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c^2}\,\frac{\vec{v}(t_R)}{r} \quad , \quad t_R = t - r/c \text{ retarded time.}$$

$$\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} + \frac{\hat{r}\cdot\vec{r}_0(t_R)}{r^2} + \frac{1}{c}\,\frac{\hat{r}\cdot\vec{v}(t_R)}{r}\right]$$

$$\vec{v}(s) = \frac{d\vec{r}_0(s)}{ds} \quad \text{and} \quad \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$

Also as a check, show that these potentials satisfy the Lorentz gauge condition: $\partial_\mu A^\mu = 0$.

f) Find the magnetic field:

$$\vec{B}(\vec{r},t) = \vec{\nabla} \times \vec{A}(\vec{r},t)$$

$$= \frac{q}{4\pi\epsilon_0 c} \left\{ \frac{\vec{v}(t_R) \times \hat{\underline{r}}}{r^2} + \frac{1}{c} \frac{\vec{a}(t_R) \times \hat{\underline{r}}}{r} \right\}$$

where $\vec{a}(t_R)$ is the acceleration of the particle at the retarded time.

Compare this formula with the $\vec{B}$ you obtained in part (B) of prb. 1 in this HW 3.

g) Find the electric field $\vec{E}_R$ in the radiation zone:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E}_R = \lim_{r\to\infty} \vec{E} = -\frac{q}{4\pi\epsilon_0 c^2} \frac{\vec{a}_\perp(t_R)}{r}, \quad t_R = t - r/c$$

Interpret with a diagram.

h) Show that the radiation fields are transverse:

$$\hat{r}\cdot\vec{E}_R = \hat{r}\cdot\vec{B}_R = \vec{E}_R\cdot\vec{B}_R = 0,$$

and $$c\vec{B}_R = \hat{r}\times\vec{E}_R$$

i) Compute the average energy radiated by the electric dipole (averaged over one cycle) per unit time and per unit solid angle, and also the total average energy radiated per unit time, i.e.,

$$\frac{dP}{d\Omega} \quad \text{and} \quad P = \int d\Omega \frac{dP}{d\Omega}$$

j) Show that the radiation fields dominate outside a distance of order wavelength away from the source.

(Hint: Compare the magnitudes of two terms in $\vec{B}$ found in part f above).