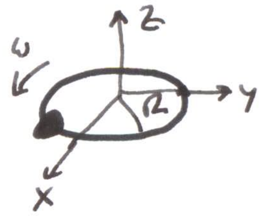
HW4-1

Phys. 402

Spring 2010

Homework 4

1) A particle of charge q moves in a circular path of radius R in the x-y plane with constant angular velocity ω_0.



In class we have shown that

$$\frac{\partial^2 P(T)}{\partial \omega \, \partial \Omega} = \sum_{n=1}^{\infty} \delta(\omega - n\omega_0) \frac{\partial P_n(T)}{\partial \Omega}$$

and

$$\frac{dP_n(T)}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{q^2 n^2 \omega_0^2}{2\pi c} \left[\frac{\beta^2}{2}\left(J_{n-1}^2 + J_{n+1}^2\right) - J_n^2\right]$$

where all the Bessels have the argument

$$J\left(\frac{\omega R}{c}\sin\theta\right) = J\left(\frac{n\omega_0 R}{c}\sin\theta\right) = J\left(n\beta\sin\theta\right)$$

$$\beta = \frac{\omega_0 R}{c}$$

HW4-2

a) Use the Bessel recursion relations

$$J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x) \qquad (b1)$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \qquad (b2)$$

and show that

$$2\left(J_{n-1}^2 + J_{n+1}^2\right) = 4\left(J_n'^2 + \frac{n^2}{x^2} J_n^2\right)$$

and also

$$\frac{\beta^2}{2}\left(J_{n-1}^2 + J_{n+1}^2\right) - J_n^2 = \beta^2 J_n'^2 + J_n^2 \frac{1}{\tan^2\theta}$$

Thus show that

$$\frac{dP_n(T)}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{n^2 \omega_0^4 \, q^2 R^2}{2\pi c^3}\right)$$

$$\times \left\{ J_n'^2(n\beta\sin\theta) + \frac{J_n^2(n\beta\sin\theta)}{\beta^2\tan^2\theta} \right\}$$

b) Show that in the non-relativistic limit* $(\beta \to 0)$ the average power radiated is all in the fundamental,

$$\frac{d P_{n=1}(T)}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{\omega_0^4}{8\pi c^3} q^2 R^2 \left(1+\cos^2\theta\right)$$

Compare this result with the rotating dipole problem Eq. 49 of the Electric Dipole Radiation (Phys. 202) notes $(qR = |p_0|)$.

(* As $x \to 0$, $J_n(x) \to \dfrac{x^n}{2^n n!}$) (b3)

HW4_4

2) A point charge q moves in a simple harmonic motion along the z-axis,

$$\vec{r}_0(t') = D\cos\omega_0 t' \, \hat{z}$$

a) Use the Lienard-Wiechert fields and show that the instantaneous power radiated per unit solid angle in terms of charge's own time t' is:

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi\epsilon_0}\frac{c\beta^4}{4\pi D^2}\frac{\sin^2\theta\,\cos^2\omega_0 t'}{(1+\beta\cos\theta\sin\omega_0 t')^5}$$

where $\beta = \frac{D\omega_0}{c}$

b) Time averaged power per unit steradian

$$\frac{d\bar{P}}{d\Omega} = \frac{1}{T}\int_0^T \frac{dP(t')}{d\Omega}dt' = \frac{q^2}{4\pi\epsilon_0}\frac{c\beta^4}{32\pi D^2}\left[\frac{4+\beta^2\cos^2\theta}{(1-\beta^2\cos^2\theta)^{7/2}}\right]\sin^2\theta$$

$$T = 2\pi/\omega_0$$

Do not take the integral. But discuss how you can do it by contour integration. Make a plot (use MATLAB) of the angular part for $\beta^2 = 0.1$, $\beta^2 = 0.5$, $\beta^2 = 0.98$

c) The linear charge density is

$$\rho(z,t) = q\,\delta(z - D\cos\omega_0 t)$$

Show that since $\rho(z,t) = \rho(z,t+T)$, periodic

it can be expanded in a Fourier series:

$$\rho(z,t) = \sum_{n=-\infty}^{+\infty} \rho(z,\omega_n)\, e^{-i\omega_n t} \quad , \quad \omega_n = n\omega_0$$

Obtain the Fourier coefficient $\rho(z,\omega_n)$

$$\rho(z,\omega_n) = \frac{1}{T}\int_0^T dt\, e^{i\omega_n t}\rho(z,t)$$

$$= \frac{q}{D}\frac{1}{i^n}\tilde{J}_n(z/D)$$

where $\tilde{J}_n(a)$ is the Fourier component of the Bessel function J_n:

$$\tilde{J}_n(a) = \int d\lambda\, e^{ia\lambda} J_n(\lambda)$$

d) Find the Fourier components of the current density $\vec{j}(z,t)$ and show that

$$\vec{j}(z,t) = \sum_{n=-\infty}^{+\infty} \vec{j}(z,\omega_n)\, e^{-i\omega_n t}$$

where $\quad j(z,\omega_n) = \frac{1}{T}\int_0^T dt\, e^{i\omega_n t}\, j(z,t)$

$$= \frac{q\,\omega_n}{i^n}\, I_n(z/D)$$

where $I_n(a) \equiv \int d\lambda\, e^{ia\lambda}\, \frac{J_n(\lambda)}{\lambda}$

e) Check the equation of continuity among the Fourier components of the charge and current densities:

$$i\omega_n\, \rho_n(z,\omega_n) \overset{?}{=} \frac{d}{dz} j(z,\omega_n)$$

f) Start from the vector potential in the Lorentz gauge

$$\vec{A}(\vec{r},t) = \int d\vec{r}\,'\int dt'\, G_{Ret}(\vec{r}t,\vec{r}\,'t')\,\vec{j}(\vec{r}\,',t')$$

(Here $\vec{J}(\vec{r}\,',t') = \vec{v}(t')\,q\,\delta(x')\,\delta(y')\,\delta(z' - D\cos\omega_0 t')$)

and obtain, for $\vec{r}$ in the radiation zone:

$$\vec{A}_R(\vec{r},t) = \sum_n A_R(\vec{r},\omega_n)\, e^{-i\omega_n t}$$

where $\vec{A}_R(\vec{r},\omega_n) = \frac{\mu_0}{4\pi} \frac{e^{ik_n r}}{r} \vec{j}(\vec{k}_n,\omega_n)$
$|\vec{r}| \to \infty$

and $k_n = nk = \frac{n\omega_0}{c} = \frac{\omega_n}{c}$ and

$$\vec{j}(\vec{k}_n,\omega_n) = \frac{1}{T}\int_0^T dt\, \vec{j}(\vec{k}_n,t)\, e^{i\omega_n t} \qquad \vec{k}_n = k_n \hat{\underline{r}} = \frac{n\omega_0}{c}\hat{\underline{r}}$$

and $\vec{j}(k_n,t) = \int d\vec{r}'\, e^{-i\vec{k}_n\cdot\vec{r}'}\, \vec{j}(\vec{r}',t)$

$$= \int d\vec{r}'\, e^{-i\vec{k}_n\cdot\vec{r}'}\, q\,\vec{v}(t)\,\delta(\vec{r}'-\vec{r}_0(t))$$

Show that

$$\vec{j}(\vec{k}_n,\omega_n) = \frac{q\omega_0 D}{i^n} \frac{J_n(n\beta\cos\theta)}{\beta\cos\theta}\hat{\underline{z}}$$

g) Show that $\vec{S}(\vec{r},t) = \frac{1}{\mu_0}\cdot\left(\sum_{n=-\infty}^{+\infty} \vec{E}(r,\omega_n)\, e^{-i\omega_n t}\right) \times \left(\sum_{m=-\infty}^{+\infty} \vec{B}_m(\omega_n)\, e^{-i\omega_m t}\right)$

when averaged over time yields

$$\langle \vec{S}(\vec{r},t)\rangle_T = \sum_{n=1,2..} \frac{2}{\mu_0} \mathrm{Re}\, \vec{E}^*(\omega_n) \times \vec{B}(\omega_n)$$

and $r^2\, \hat{\underline{r}}\cdot\langle\vec{S}\rangle = \sum_{n=1,2..}^{\infty} \frac{dP_n}{d\Omega}$

and $\frac{d\bar{P}_n}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{q^2 \omega_n^2}{2\pi c} \cdot \tan^2\theta \; J_n^2(n\beta\cos\theta)$

$n \geq 1$

$\omega_n = n\omega_0 \qquad \beta = \frac{\omega_0 D}{c}$

h) Show that in the non-relativistic limit, $(\beta \to 0)$ the average power radiated is all in the fundamental

$$\frac{d\bar{P}_{n=1}}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0}\right) \frac{\omega_0^4}{8\pi c^3} q^2 D^2 \sin^2\theta$$

Compare this result with the point dipole result given in class, or Eq. 42 of the Electric Dipole Radiation (Phys. 202) notes. $(qD = |p_0|)$