Induced Electric Fields

In chapter 25 \( \Rightarrow \Delta V = V_b - V_a = - \int_a^b \vec{E} \cdot d\vec{l} \)
when the \( \vec{E} \) is conservative, \( \oint \vec{E} \cdot d\vec{l} = 0 \).

In chapter 34 \( \Rightarrow \) changing \( \Phi_B \) generates an induced electric field.
\[ E = \oint \vec{E}_{nc} \cdot d\vec{l} \]
\[ \uparrow \text{this is a non-conservative electric field.} \]
Thus \( \oint \vec{E}_{nc} \cdot d\vec{l} \neq 0 \).

Using the Faraday's law
\[ E = -\frac{d\Phi_B}{dt} \] and \( E = \oint \vec{E}_{nc} \cdot d\vec{l} \) \( \Rightarrow \)
\[ \oint \vec{E}_{nc} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \]
changing magnetic flux will induce an electric field.

Important: Electric charges generate conservative electric fields.
Changing magnetic flux generate non-conservative electric fields.

Important: Read section 31.4 in the book.
Study the Example 31.7 on page 906.
10.9.1 Rectangular Loop Near a Wire

An infinite straight wire carries a current $I$ is placed to the left of a rectangular loop of wire with width $w$ and length $l$, as shown in the Figure 10.9.1.

![Figure 10.9.1 Rectangular loop near a wire](image)

(a) Determine the magnetic flux through the rectangular loop due to the current $I$.

(b) Suppose that the current is a function of time with $I(t) = a + bt$, where $a$ and $b$ are positive constants. What is the induced emf in the loop and the direction of the induced current?

**Solutions:**

(a) Using Ampere's law:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enc}}$$

(10.9.1)

the magnetic field due to a current-carrying wire at a distance $r$ away is

$$B = \frac{\mu_0 I}{2\pi r}$$

(10.9.2)

The total magnetic flux $\Phi_\theta$ through the loop can be obtained by summing over contributions from all differential area elements $dA = l dr$:

$$\Phi_\theta = \int d\Phi_\theta = \int \mathbf{B} \cdot d\mathbf{A} = \frac{\mu_0 I}{2\pi} \int_{s}^{s+w} \frac{dr}{r} = \frac{\mu_0 I}{2\pi} \ln \left( \frac{s+w}{s} \right)$$

(10.9.3)

Note that we have chosen the area vector to point into the page, so that $\Phi_\theta > 0$.

(b) According to Faraday's law, the induced emf is

$$\varepsilon = -\frac{d\Phi_\theta}{dt} = -\frac{d}{dt} \left[ \frac{\mu_0 I}{2\pi} \ln \left( \frac{s+w}{s} \right) \right] = -\left( \frac{\mu_0 I}{2\pi} \right) \ln \left( \frac{s+w}{s} \right) \cdot \frac{dl}{dt} = -\frac{\mu_0 b l}{2\pi} \ln \left( \frac{s+w}{s} \right)$$

(10.9.4)

where we have used $dl/dt = b$. 
The straight wire carrying a current $I$ produces a magnetic flux into the page through the rectangular loop. By Lenz’s law, the induced current in the loop must be flowing \textit{counterclockwise} in order to produce a magnetic field out of the page to counteract the increase in inward flux.

### 10.9.2 Loop Changing Area

A square loop with length $l$ on each side is placed in a uniform magnetic field pointing into the page. During a time interval $\Delta t$, the loop is pulled from its two edges and turned into a rhombus, as shown in the Figure 10.9.2. Assuming that the total resistance of the loop is $R$, find the average induced current in the loop and its direction.

![Figure 10.9.2 Conducting loop changing area](image)

**Solution:**

Using Faraday’s law, we have

$$\varepsilon = -\frac{\Delta \Phi_B}{\Delta t} = -B \left( \frac{\Delta A}{\Delta t} \right)$$  \hspace{1cm} (10.9.5)

Since the initial and the final areas of the loop are $A_i = l^2$ and $A_f = l^2 \sin \theta$, respectively (recall that the area of a parallelogram defined by two vectors $\overrightarrow{I}_1$ and $\overrightarrow{I}_2$ is $A = |\overrightarrow{I}_1 \times \overrightarrow{I}_2| = l_1 l_2 \sin \theta$), the average rate of change of area is

$$\frac{\Delta A}{\Delta t} = \frac{A_f - A_i}{\Delta t} = -\frac{l^2 (1 - \sin \theta)}{\Delta t} < 0$$  \hspace{1cm} (10.9.6)

which gives

$$\varepsilon = \frac{Bl^2 (1 - \sin \theta)}{\Delta t} > 0$$  \hspace{1cm} (10.9.7)

Thus, the average induced current is
\[ I = \frac{\varepsilon}{R} = \frac{BI^2(1 - \sin \theta)}{\Delta t R} \quad (10.9.8) \]

Since \((\Delta A / \Delta t) < 0\), the magnetic flux into the page decreases. Hence, the current flows in the clockwise direction to compensate the loss of flux.

### 10.9.3 Sliding Rod

A conducting rod of length \(l\) is free to slide on two parallel conducting bars as in Figure 10.9.3.

![Figure 10.9.3 Sliding rod](image)

In addition, two resistors \(R_1\) and \(R_2\) are connected across the ends of the bars. There is a uniform magnetic field pointing into the page. Suppose an external agent pulls the bar to the left at a constant speed \(v\). Evaluate the following quantities:

(a) The currents through both resistors;

(b) The total power delivered to the resistors;

(c) The applied force needed for the rod to maintain a constant velocity.

**Solutions:**

(a) The emf induced between the ends of the moving rod is

\[ \varepsilon = -\frac{d \Phi_n}{dt} = -Blv \quad (10.9.9) \]

The currents through the resistors are

\[ I_1 = \frac{|\varepsilon|}{R_1}, \quad I_2 = \frac{|\varepsilon|}{R_2} \quad (10.9.10) \]

Since the flux into the page for the left loop is decreasing, \(I_1\) flows clockwise to produce a magnetic field pointing into the page. On the other hand, the flux into the page for the right loop is increasing. To compensate the change, according to Lenz’s law, \(I_2\) must flow counterclockwise to produce a magnetic field pointing out of the page.

(b) The total power dissipated in the two resistors is
\[ P_R = I_1 |\mathcal{E}| + I_2 |\mathcal{E}| = (I_1 + I_2) |\mathcal{E}| = \mathcal{E}^2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = B^2 l^2 v^2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \] (10.9.11)

(c) The total current flowing through the rod is \( I = I_1 + I_2 \). Thus, the magnetic force acting on the rod is

\[ F_B = II'B = |\mathcal{E}| l B \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = B^2 l^2 v \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \] (10.9.12)

and the direction is to the right. Thus, an external agent must apply an equal but opposite force \( \mathbf{F}_{\text{ext}} = -\mathbf{F}_B \) to the left in order to maintain a constant speed.

Alternatively, we note that since the power dissipated in the resistors must be equal to \( P_{\text{ext}} \), the mechanical power supplied by the external agent. The same result is obtained since

\[ P_{\text{ext}} = \mathbf{F}_{\text{ext}} \cdot \mathbf{v} = F_{\text{ext}} v \] (10.9.13)
10.9.5 Time-Varying Magnetic Field

A circular loop of wire of radius $a$ is placed in a uniform magnetic field, with the plane of the loop perpendicular to the direction of the field, as shown in Figure 10.9.5.

![Figure 10.9.5 Circular loop in a time-varying magnetic field](image)

The magnetic field varies with time according to $B(t) = B_0 + bt$, where $B_0$ and $b$ are positive constants.

(a) Calculate the magnetic flux through the loop at $t = 0$.

(b) Calculate the induced emf in the loop.

(c) What is the induced current and its direction of flow if the overall resistance of the loop is $R$?

(d) Find the power dissipated due to the resistance of the loop.

**Solution:**

(a) The magnetic flux at time $t$ is given by
\[ \Phi_B = BA = (B_0 + bt)(\pi a^2) = \pi (B_0 + bt)a^2 \quad (10.9.17) \]

where we have chosen the area vector to point into the page, so that \( \Phi_B > 0 \). At \( t = 0 \), we have

\[ \Phi_B = \pi B_0 a^2 \quad (10.9.18) \]

(b) Using Faraday’s Law, the induced emf is

\[ \varepsilon = -\frac{d\Phi_B}{dt} = -A \frac{dB}{dt} = -(\pi a^2) \frac{d (B_0 + bt)}{dt} = -\pi ba^2 \quad (10.9.19) \]

(c) The induced current is

\[ I = \frac{|\varepsilon|}{R} = \frac{\pi ba^2}{R} \quad (10.9.20) \]

and its direction is counterclockwise by Lenz’s law.

(d) The power dissipated due to the resistance \( R \) is

\[ P = I^2 R = \left( \frac{\pi ba^2}{R} \right)^2 R = \frac{(\pi ba^2)^2}{R} \quad (10.9.21) \]

10.9.6 Moving Loop

A rectangular loop of dimensions \( l \) and \( w \) moves with a constant velocity \( \vec{v} \) away from an infinitely long straight wire carrying a current \( I \) in the plane of the loop, as shown in Figure 10.9.6. Let the total resistance of the loop be \( R \). What is the current in the loop at the instant the near side is a distance \( r \) from the wire?

![Figure 10.9.6 A rectangular loop moving away from a current-carrying wire](image)

**Solution:**

The magnetic field at a distance \( s \) from the straight wire is, using Ampere’s law:
\[ B = \frac{\mu_0 I}{2\pi s} \quad (10.9.22) \]

The magnetic flux through a differential area element \( dA = lds \) of the loop is

\[ d\Phi_B = \vec{B} \cdot d\vec{A} = \frac{\mu_0 I}{2\pi s} l ds \quad (10.9.23) \]

where we have chosen the area vector to point \textit{into} the page, so that \( \Phi_B > 0 \). Integrating over the entire area of the loop, the total flux is

\[ \Phi_B = \frac{\mu_0 I l}{2\pi} \int_r^{r+w} ds = \frac{\mu_0 I l}{2\pi} \ln \left( \frac{r+w}{r} \right) \quad (10.9.24) \]

Differentiating with respect to \( t \), we obtain the induced emf as

\[ \varepsilon = -\frac{d\Phi_B}{dt} = -\frac{\mu_0 I l}{2\pi} \frac{dr}{dt} \left( \ln \frac{r+w}{r} \right) = -\frac{\mu_0 I l}{2\pi} \left( \frac{1}{r+w} - \frac{1}{r} \right) \frac{dr}{dt} = \frac{\mu_0 I l}{2\pi} \frac{vw}{r(r+w)} \quad (10.9.25) \]

where \( v = \frac{dr}{dt} \). Notice that the induced emf can also be obtained by using Eq. (10.2.2):

\[ \varepsilon = \oint (\vec{v} \times \vec{B}) \cdot d\vec{s} = vl \left[ B(r) - B(r+w) \right] = vl \left[ \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I}{2\pi (r+w)} \right] = \frac{\mu_0 I l}{2\pi} \frac{vw}{r(r+w)} \quad (10.9.26) \]

The induced current is

\[ I = \frac{\varepsilon}{R} = \frac{\mu_0 I l}{2\pi R} \frac{vw}{r(r+w)} \quad (10.9.27) \]

### 10.10 Conceptual Questions

1. A bar magnet falls through a circular loop, as shown in Figure 10.10.1
(a) Describe qualitatively the change in magnetic flux through the loop when the bar magnet is above and below the loop.

(b) Make a qualitative sketch of the graph of the induced current in the loop as a function of time, choosing $I$ to be positive when its direction is counterclockwise as viewed from above.

2. Two circular loops $A$ and $B$ have their planes parallel to each other, as shown in Figure 10.10.2.

Loop $A$ has a current moving in the counterclockwise direction, viewed from above.

(a) If the current in loop $A$ decreases with time, what is the direction of the induced current in loop $B$? Will the two loops attract or repel each other?

(b) If the current in loop $A$ increases with time, what is the direction of the induced current in loop $B$? Will the two loops attract or repel each other?

3. A spherical conducting shell is placed in a time-varying magnetic field. Is there an induced current along the equator?

4. A rectangular loop moves across a uniform magnetic field but the induced current is zero. How is this possible?
A rectangular loop of wire with mass $m$, width $w$, vertical length $l$, and resistance $R$ falls out of a magnetic field under the influence of gravity, as shown in Figure 10.11.11. The magnetic field is uniform and out of the paper ($\vec{B} = B\hat{\mathbf{i}}$) within the area shown and zero outside of that area. At the time shown in the sketch, the loop is exiting the magnetic field at speed $\vec{v} = -v\hat{\mathbf{k}}$.

(a) What is the direction of the current flowing in the circuit at the time shown, clockwise or counterclockwise? Why did you pick this direction?

(b) Using Faraday's law, find an expression for the magnitude of the emf in this circuit in terms of the quantities given. What is the magnitude of the current flowing in the circuit at the time shown?

(c) Besides gravity, what other force acts on the loop in the $\pm \hat{\mathbf{k}}$ direction? Give its magnitude and direction in terms of the quantities given.

(d) Assume that the loop has reached a “terminal velocity” and is no longer accelerating. What is the magnitude of that terminal velocity in terms of given quantities?

(e) Show that at terminal velocity, the rate at which gravity is doing work on the loop is equal to the rate at which energy is being dissipated in the loop through Joule heating.
10.11.3 RC Circuit in a Magnetic Field

Consider a circular loop of wire of radius $r$ lying in the $xy$ plane, as shown in Figure 10.11.3. The loop contains a resistor $R$ and a capacitor $C$, and is placed in a uniform magnetic field which points into the page and decreases at a rate $dB/dt = -\alpha$, with $\alpha > 0$.

![Diagram of an RC circuit in a magnetic field](image)

**Figure 10.11.3 RC circuit in a magnetic field**

(a) Find the maximum amount of charge on the capacitor.

(b) Which plate, $a$ or $b$, has a higher potential? What causes charges to separate?