AN INTRODUCTION TO INTEGRATION
OVER CURVES AND SURFACES FOR
STUDENTS OF ELECTRICITY AND
MAGNETISM

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Dedicated to Dee Aynn Mary Dye Johnson
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Preface

Calculus-based, general-physics courses on electricity and magnetism for physical science and engineering majors make extensive use of some advanced integral-calculus concepts known as line integrals, double integrals, and surface integrals. Indeed, of the six fundamental laws of physics taught in the course (Coulombs law, the law of Biot-Savart, Gauss law for Electricity, Gauss law for magnetism, Faradays law and the Ampere-Maxwell law), all of them except the first one listed are usually stated in terms of, either a line integral, a surface integral, or both. Even the exception (Coulombs law) involves line and surface integrals when it is applied to study the electrical forces exerted by charges that are distributed respectively over lines and surfaces. Moreover, the important concept of electrical potential (voltage) is defined in terms of a special type of line integral, sometimes called work integrals. Consequently, students in these courses cannot understand the subject material with any depth without having an understanding of these advanced integrals. Furthermore, they cannot apply that subject material to solve mathematical-physics and engineering problems without knowing how to work with and evaluate such integrals.

The subject of line, double, and surface integrals is not covered in the standard physics textbooks that could be used as textbooks for the electricity and magnetism course. In particular, consider the textbook Halliday, Resnick, and Walker[HRW08], which is the textbook I use in the physics courses I teach at Fullerton College. This is the longtime standard physics text used by many colleges. It uses line and surface integrals freely and frequently in the sections dealing with electricity and magnetism, apparently under the assumption that the readers have studied these integrals elsewhere.

The theory of these advanced-level integrals are taught in most mathematics departments during the third semester of calculus in a subject known as Multivariable Calculus. Many colleges and universities, however, do not require a course in multivariable calculus as a prerequisite for taking the physics course. Indeed, the math prerequisite requirement for the calculus-based course on electricity and magnetism at Fullerton College demands only completion or concurrent enrollment in the second semester of calculus. This choice is made in order to make it possible for engineering and physical science majors to proceed in a more timely manner with the study of the physics needed for their major. Consequently, the subject of line and surface integrals must be taught in the physics course itself. At Fullerton College, this is made possible by reducing
the number of physics-textbook chapters covered in the course from the usual fifteen to twelve, freeing up approximately three to four weeks of class time to be used for the study of the required mathematics.

Having a good textbook for all of the material being covered in a course is important because it makes it possible to reduce the amount of lecture time. Physics-education research has shown us that replacing a significant amount of lecture time with activities that actively involve the students minds with the subject material pays big dividends with more students achieving the desired student-learning outcomes. In my classes, I make substantial use of the textbook for this purpose and believe that it has been very beneficial. I have even made all of my exams open book so I can concentrate on assessing student ability to delve more deeply into the subject material and apply it to novel situations that the student hasn’t seen before.

Unfortunately, however, there is no textbook available that serves the needs of the mathematical portion of the electricity and magnetism course. Certainly, there are excellent treatments of the line and surface integrals in mathematics texts on Multivariable Calculus, but, in each book, that material is limited to about 20% of the book, making it unreasonable to require the students to buy one. Moreover, in those books, the discussion of the material important to us makes frequent use of material taught earlier in the book, which is not needed by the students in the electricity and magnetism course. This requires the students to study considerable material that is extraneous to the physics course.

In order for a book to satisfy the above mentioned need, it must meet several conditions. First, it must treat all of the portion of the theory of multivariable calculus that is needed by the students in the physics course. Secondly, it must be short enough to be taught well within the 4-unit physics course in three to four weeks. Third, it should include clearly written sample problems, which are illustrated with well-drawn figures and are solved in detail to illustrate the application of the theory. Finally, it should include a number of exercises for the students to solve to help them develop the skills required to apply the theory.

The present book is designed specifically to meet the mathematical needs of the students taking the calculus-based electricity and magnetism course. In order to satisfy the first two, rather conflicting, conditions listed above, the choice of the topics included and the way they are presented had to be tailored very carefully to the demands of the course. Moreover, the motivations given for the topics and many of the examples are directed specifically to students currently studying electricity and magnetism. The result is a rather novel text, quite different from anything else currently available.

The mathematical material in the book is placed in the order that it is normally encountered in the study of electricity and magnetism. In the physics course, I have found it to be beneficial to integrate the mathematics portion of the course in with the physics so that each new mathematics concept is introduced shortly before it is needed for the first time to discuss the physics. An illustration of how this is done in my Electricity and Magnetism course at Fullerton College is given in Table 1. The course covers 16 weeks with three units of lecture and one unit of laboratory integrated together in two 3-hour
classes each week. Conceptual and analytic exams are given every week, so there are no midterm exams. The final exam is given during an additional final-exam week. In the table, the notation “HRW” refers to the physics textbook Halliday, Resnick, and Walker [HRW08] whereas “Sherman” refers to this mathematics textbook [She09]. The number that follows the name HRW or Sherman indicates the number of the chapter in the corresponding book that is covered in that unit. Each unit listed in the table (except Units 1 and 16) includes a reading assignment of the chapters listed, conceptual homework, lecture, conceptual exam, group project, analytic homework, analytic exam, and Laboratory.

<table>
<thead>
<tr>
<th>Unit No.</th>
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Sherman 1. Introduction  
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| 3        | 1.5          | Sherman 3. Charge Integrals  
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HRW 24. Electric Potential |
| 9        | 1            | HRW 25. Capacitance |
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| 12       | 1            | HRW 28. Magnetic Fields |
| 13       | 1            | HRW 29. Magnetic Fields Due to Currents |
| 14       | 1            | HRW 30. Induction and Inductance |
| 15       | 1            | HRW 31. Electromagnetic Oscillations |
| 16       | .5           | HRW 32. Maxwell’s Equations |

Although the instruction using the mathematics textbook is completed by the end of Unit 8, the material covered in the book continues to be used heavily throughout Units 12 through 16.

In conclusion, some comments about the mathematics course of Multivariable Calculus are in order. The fact that the use of this textbook in the electricity and magnetism course makes it possible for students to take that course without having to take Multivariable Calculus should not be taken to mean that physics/engineering students can avoid taking the latter course. This book covers less than 20% of the material taught in the mathematics course, and physics/engineering students need to learn all of the material taught there.
There is no need to be concerned about the overlap of material when you take both the physics and the math class. You will find that the overlap is truly useful in helping you to gain a working knowledge of the material. The order in which you take the two classes is not very important. You can do it either way. If it doesn’t put you too far behind in your physics/engineering studies, however, you will probably find it beneficial to take the mathematics course first. Then you won’t have to struggle so hard with the mathematics while you are struggling with the physics. Do not assume, however, that taking the mathematics course first will enable you to take the physics course without having to study this text. Students who take my course after having already studied Multivariable Calculus, invariably tell me that this textbook is still very useful to them because of it’s focus on the needs of the physics students. The book helps them to learn how to apply what they learned in the mathematics class to increase their understanding of the physics material.

I wrote this book while on sabbatical leave from Fullerton College for that purpose during the Fall Semester of 2006. It is a pleasure to acknowledge the support of North Orange County Community College District during that time. I am grateful to many of my colleagues at Fullerton College and 15 other colleges and universities for making written recommendations to my district that it support me in this endeavor. I especially want to thank Sandro Corsi in the Art Computer Graphics Department at Fullerton College, who spent many hours helping me to learn the details about how to use Adobe Illustrator to make the figures for the book. Moreover, I would like to thank my son Wayne Sherman, a Professor of Mathematics at San Diego Miramar College, who served as my mathematical consultant while I was writing the book. Finally, I wish to acknowledge the hard work of graphic designer Katherine Tyler, who, on very short notice, finalized my hand-drawn figures in digital form using Adobe Illustrator, just in time to be included in the manuscript.

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Chapter 1

Introduction

This chapter gives a brief overview of the book and discusses how its topics are related to the various subjects in the study of electricity and magnetism. Do not be concerned if you don’t understand this discussion with any depth. That cannot be done until after you have learned much of the material being taught in the course. For example, electric and magnetic fields are mentioned here, even though you will not know what they are until later in the course. Just try to get a broad picture of how the course is going to flow from one topic to another without knowing the details of what those topics are about. Then, feel free to revisit this chapter from time to time as you progress in the course. It could help you to see how the various topics in the course are interrelated.

The primary subject of study in the electricity and magnetism course concerns the properties of electric and magnetic fields, including how they are produced, how they interact with each other, and what forces they exert on electric charges and currents. Apart from the forces exerted by the fields on charges and currents, the above mentioned properties are governed by four fundamental laws, which are described by four equations known collectively as Maxwell’s equations. These equations can be expressed in two different forms, the differential form (in terms of derivatives) and the integral form (in terms of integrals). Most textbooks for calculus-based fundamental physics courses present only the integral form because Maxwell’s equations are most suitable in that form for deriving many of the basic results taught in the introductory course.

Unfortunately, the integrals involved are not the simple definite and indefinite integrals that are taught in the first and second semester of calculus. Rather, they are more advanced forms of integrals, taught in the third semester of calculus in a course called Multivariable Calculus. These advanced integrals are the subject of the present book. To be more specific, the types of integrals discussed herein are as follows (presented in the order they are treated in this book):

a. charge integrals,
b. line integrals,
CHAPTER 1. INTRODUCTION

c. double (area) integrals,
d. surface integrals,
e. flux integrals,
f. work integrals,
g. line integrals involving a vector product,
h. triple (volume) integrals.¹

Except for the discussion of triple integrals, which is presented in an appendix at the end, these topics are presented in the book in roughly the same order as they typically appear in a fundamental-physics course in electricity and magnetism. This makes it possible for the book to be integrated into the physics course in a straightforward manner.

Each of the above listed integrals is defined by extending the definition of the basic definite integral in some small way. Consequently, if you understand the concepts involved in the definition of the definite integral, you will be able to easily grasp the definition of each new integral type. To help refresh your memory, we begin with a short chapter (Chapter 2) which presents a brief review of the definite integral. You may find it useful to refer back to that chapter each time you begin the study of a new integral type in a new chapter. You should find the chapter useful even if you have a good understanding of the meaning of the definite integral, because the presentation there emphasizes the concepts that are used in later chapters to define the new integrals.

Understanding the definition of (and therefore, the meaning of) each type of integral is important for the electricity and magnetism course, but that is only half the battle. In addition, we must be able to evaluate integrals of each type. Recall that the definition of definite integrals is rarely used for evaluating the integral. Usually, we use a very different approach employing the Fundamental Theorem of Calculus for that purpose. Likewise, the definitions of the new integrals are not very useful for evaluating them. For this reason, each chapter discusses how to evaluate the type of integrals it is treating. This always involves converting the type of integral we are dealing with into one or more definite integrals so that we can apply the standard methods for evaluating definite integrals.

The concept of “charge integrals” is encountered near the beginning of the electricity and magnetism course after the presentation of Coulomb’s law. Coulomb’s law tells us the magnitude of the electrical force felt by one point charge in the presence of a second point charge at another location. With slight modification, it can also be used to determine the electric field produced at a point in space by a point charge located elsewhere. Coulomb’s law can not be used alone, however, to determine the electric field produced at a point in space by charge that is distributed over an extended region of space. For that purpose, we must add the concept of charge integrals and integrate Coulomb’s law over the charge distribution. This is the topic of Chapter 3.

¹Triple integrals are not nearly so crucial to the theory of electricity and magnetism as are the integrals listed above. Nonetheless, they are needed occasionally in the theory. For this reason, a brief summary of the properties of triple integrals is given in an appendix.
Except in one special case, Chapter 3 doesn’t explain how to evaluate charge integrals. It does, however, make an important step in that direction. It shows how to transform a charge integral into a integral over the region of space that is occupied by the charge distribution. The resulting integrals can be called space integrals. The one special case in which the chapter shows how to evaluate the integral is when the charge is distributed along a straight line segment which can be taken to be a section of the x axis. In that case, the space integral is simply a definite integral over that section of the x axis.

If the charge is distributed along a straight or curved line, the resulting space integral belongs to a category of integrals known as “line integrals”, discussed in Chapter 4. If the charge is distributed over a curved or flat surface, the resulting integral belongs to the category known as “surface integrals” discussed in Chapter 6. If the surface is flat, the space integral also belongs to simpler category known as “double integrals” discussed in Chapter 5. If the charge is distributed over a three-dimensional volume, the space integral belongs to a category known as triple integrals discussed in Appendix A. Each of these discussions explains how to convert the relevant space integral into definite integrals which can be evaluated by standard methods.

Apart from triple integrals, all of these categories of integrals have important applications throughout the subject of electricity and magnetism beyond the application involving charge integrals. In particular, all four of Maxwell’s equations involve a special case of surface integrals known as “flux integrals” discussed in Section 6.5. Moreover, two of Maxwell’s equations (Faraday’s law and the Ampere-Maxwell law) both involve a special case of line integrals known as “work integrals” discussed in Section 7.3. Work integrals are important, also, in the definition of electric potential (voltage). The last section of Chapter 7 discusses a type of line integrals that is frequently required in the study of magnetism. In particular, these integrals (which involve the vector product of vectors) are needed to determine the force exerted by a magnetic field on a current-carrying wire when either the wire is curved or the magnetic field changes with position along the wire. This type of integral also occurs in the law of Biot and Savart which determines the magnetic field produced by a current-carrying wire.

Since triple integrals are not involved directly in Maxwell’s equations, they are not nearly as important to the theory of electricity and magnetism as the integrals mentioned above. They are needed occasionally, however, to evaluate charge integrals for charges distributed over three-dimensional volumes to determine the total charge in the volume or to determine the electric field produced by the volume charge. A brief summary of triple integrals is presented in Appendix A.

Although this book concerns mathematical topics, it is written in the loose language of a physicist who is more concerned with explaining the basic concepts than with being precise about the conditions under which those concepts are applicable. Unless stated otherwise, the results given herein are applicable as stated if the functions being integrated are continuous over the region of integration, and the regions of integration themselves are finite and sufficiently well
behaved. For example, line integrals should be over smooth curves (like a circle) which do not change directions abruptly (like a square). We can, however, apply these results to more troublesome cases by using standard techniques. For example, the line integral around a square can be done easily by breaking up the square into the sum of four smooth curves (the straight sides) and integrating along each of them separately. Functions that are piecewise continuous can be dealt with in the same way. Functions with isolated singularities in the region of integration can sometimes be integrated by excluding the neighborhood of the singularity from the region of integration when integrating and then taking the limit of the neighborhood tending to zero after completion of the integration. Some singularities of an integrand can be removed by an appropriate change of variables of integration. Some of these techniques are illustrated in the sample problems when the troublesome features arise.
Chapter 2

Review of Definite Integrals

2.1 Introduction

A very brief review of the definite integral is given in this chapter. You should remember all of this material from your introduction to integration in your first calculus class. It is essential that you have a good understanding of the concepts involved, because we are going to use them continuously throughout the course with only minor modifications to define all of the new integrals. So, if the concepts are not clear in your mind, be sure to review them in your introductory calculus book, which explains them in much more detail and illustrates them with many figures and examples. Once you understand this definition of definite integrals clearly, you should have no difficulty understanding the definitions of all of the other integrals we encounter.

The definite integral can be written

\[ I = \int_{a}^{b} f(x) \, dx. \] (2.1)

Here, the letter \( I \) on the left-hand-side of the equation is being used to represent the integral of interest. So, Eqn. (2.1) states that the integral of interest \( I \) is equal to the definite integral of the function \( f(x) \) with respect to \( x \) from the value \( x = a \) to the value \( x = b \). Another way to describe the right-hand-side of the equation is to say that it is the integral of \( f(x) \) along the \( x \) axis from \( x = a \) to \( x = b \). To give these statements meaning, we need to specify what we mean by the statement “definite integral of \( f(x) \)”. To this end, we use an operational definition. The definite integral of \( f(x) \) is the result of the process of integrating \( f(x) \) with respect to \( x \) over the interval from \( a \) to \( b \).

In the discussion of the process of integration, we distinguish between two different processes which yield exactly the same result for the integral. Both processes are referred to as integration, but they are very different operations to do in practice. Although they are mathematically equivalent, one of them is considered in this book to be the definition of the process of integration, and the
other one is considered to be the process normally used to evaluate the integral. In both cases, we assume that the function \( f(x) \) is continuous over the interval from \( a \) to \( b \).

### 2.2 Definition

The definition of the process of integration for the definite integral given above consists of several steps.

1. Partition the interval between \( a \) and \( b \) along the \( x \) axis into \( n \) tiny sub-intervals \( \Delta x_i \), where \( i \) is an integer that runs from \( i = 1 \) to \( i = n \). The \( n \) tiny sub-intervals are chosen so that all of them together exactly cover the interval from \( a \) to \( b \), and they all decrease in size simultaneously to zero as the number \( n \) increases to infinity.

2. Multiply the size of each tiny sub-interval by the value of the function \( f(x) \) evaluated at the center \( x_i \) of that tiny interval.

3. Add together all \( n \) of the above products of the multiplications.

4. Take the limit of the sum as the number \( n \) of tiny sub-intervals tends to infinity.

The result of this integration process is the integral \( I \) given in Eqn. (2.1). After doing the first step described above, all of the remaining steps can be expressed mathematically by a single equation

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i.
\]  

(2.2)

Equation (2.2), taken together with the subdivision of the interval from \( a \) to \( b \) into the \( n \) tiny intervals \( \Delta x_i \), gives the mathematical definition of the definite integral. When \( b \) is greater than \( a \), we say that we are integrating in the positive \( x \) direction. In this case, the tiny sub-intervals \( \Delta x_i = (x_i)_{\text{final}} - (x_i)_{\text{initial}} \) are positive since \( (x_i)_{\text{final}} \) is larger than \( (x_i)_{\text{initial}} \). Similarly, when \( a \) is greater than \( b \), we say that we are integrating in the negative \( x \) direction, and \( \Delta x_i \) is negative. Consequently, it follows from the definition of the definite integral given in Eqn. (2.2) that

\[
\int_{a}^{b} f(x) \, dx = - \int_{b}^{a} f(x) \, dx
\]  

(2.3)

regardless of whether \( a \) is larger or smaller than \( b \). Expressing this result in words, we can say that reversing the direction of integration, reverses the sign of the integral.
2.3 Evaluation

While the above definition of the integration process is very helpful for understanding the concept of integration, it is not very useful for evaluating definite integrals. Instead, the evaluation is usually done by finding an antiderivative $F(x)$ of the function $f(x)$. The antiderivatives are defined over the region from $a$ to $b$ by

$$\frac{dF(x)}{dx} = f(x). \quad (2.4)$$

Then, according to the Fundamental Theorem of Calculus[ABD05, Sec. 6.6], the definite integral $I$ defined by Eqn. (2.1) is equal to

$$I = F(b) - F(a). \quad (2.5)$$

While it may not be easy to find an antiderivative of $f(x)$, once we find one, the evaluation of the definite integral becomes trivial using Eqn. (2.5).

**Sample Problem 2.1** Evaluate the definite integral of the function $f(x) = Ax^p$ over the interval from $x = x_1$ to $x = x_2$ where $A, p, x_1$ and $x_2$ are given real constants with $p$ not equal to $-1$.

**Solution:** Our integral of interest is

$$I = \int_{x_1}^{x_2} Ax^p \, dx. \quad (2.6)$$

If we choose to use the definition of the definite integral given in Eqn. (2.2), we obtain

$$I = \lim_{n \to \infty} \sum_{i=1}^{n} A x_i^p \Delta_i x. \quad (2.7)$$

Since $A$ can be factored outside of the sum, we see that this approach readily yields the result

$$I = \int_{x_1}^{x_2} Ax^p \, dx = A \int_{x_1}^{x_2} x^p \, dx. \quad (2.8)$$

It does not, however, readily lead to the evaluation of the integral because it is not obvious how to evaluate the sum for arbitrary integer $n$.

If we choose to use the Fundamental Theorem of Calculus as given in Eqn. (2.5), then we need to find an antiderivative $F(x)$ of $f(x)$ as defined by Eqn. (2.4). This is easy to do. Since we know that

$$\frac{dx^q}{dx} = qx^{q-1} \quad (2.9)$$

for arbitrary constant $q$, it is easy to see that

$$\frac{d}{dx} \frac{Ax^{p+1}}{p+1} = Ax^p = f(x), \quad (2.10)$$
for arbitrary $p$ not equal to -1. This shows that an antiderivative of $f(x)$ is

$$F(x) = \frac{Ax^{p+1}}{p+1}$$

for $p$ not equal to $-1$. Finally, we substitute this result into the Fundamental Theorem of Calculus as given in Eqn. (2.5) to obtain the final result

$$I = \int_{x_1}^{x_2} Ax^p \, dx = \frac{Ax_2^{p+1}}{p+1} - \frac{Ax_1^{p+1}}{p+1}$$

(2.12)

for arbitrary $p$ not equal to $-1$. If you do not have this important result memorized, you should memorize it now, because you will need it in numerous homework and exam problems throughout the course.

### 2.4 Integration By Substitution

Evaluation of definite integrals by applying the Fundamental Theorem of Calculus directly involves guessing what the antiderivative is and then using differentiation to check to see if the guess was correct. Since this is often unproductive, a number of techniques for changing an integral into another form, which may be easier to evaluate, have been developed. The technique we will find to be most useful in this course is known as integration by substitution. Two different approaches, which are mathematically equivalent but are somewhat different in practice, are reviewed in this section.

#### 2.4.1 u-Substitution

This method is useful when the integral of interest $I$ can be put in the form

$$I = \int_a^b F(g(x)) \frac{dg}{dx} \, dx,$$

(2.13)

where $g(x)$ is a function of $x$, continuous over the integration interval from $a$ to $b$, and $F(g(x))$ is a function of $g(x)$, continuous over the region that $g(x)$ ranges as $x$ varies over the same integration interval[ABDO5, Sec. 6.8]. We then make the u-substitution $u = g(x)$ and $du = \frac{dg}{dx} \, dx$ in the integral to obtain

$$I = \int_{g(a)}^{g(b)} F(u) \, du.$$  

(2.14)

**Sample Problem 2.2** Let the integral of interest be

$$I = \int_a^b (x^2 + \xi^2)^p \, 2x \, dx,$$

(2.15)

where $p$ is an arbitrary constant not equal to $-1$. 

2.4. INTEGRATION BY SUBSTITUTION

Solution: Since $2x$ is the derivative of $x^2 + \xi^2$ with respect to $x$, the above integral is in the form of the integral in Eqn. (2.13) with $g(x) = x^2 + \xi^2$. Consequently, when we make the substitution $u(x) = x^2 + \xi^2$, we obtain

$$I = \int_{u(a)}^{u(b)} u^p \, du = \frac{u(b)^{p+1} - u(a)^{p+1}}{p+1} = \frac{(b^2 + \xi^2)^{p+1} - (a^2 + \xi^2)^{p+1}}{p+1}. \quad (2.16)$$

We will have the opportunity to use this result many times throughout the course.

2.4.2 Change of Variable of Integration

In this approach, we start with the original form of the integral of interest

$$I = \int_a^b f(x) \, dx$$

and make the change of variable of integration $x = h(u), \, dx = \frac{dh}{du} \, du$, where $h(u)$ is a continuous function of the new variable of integration $u$. We can choose the function $h(u)$ to be whatever we want it to be, but this choice then determines what $dx$ is. Substitution of these equations into the integral gives

$$I = \int_{u(a)}^{u(b)} f(h(u)) \frac{dh}{du} \, du, \quad (2.17)$$

where $u = u(x)$ is the inverse of $x = h(u)$.\(^1\)

Sample Problem 2.3 Let the integral of interest be

$$I = \int_{-a}^{a} \sqrt{a^2 - x^2} \, dx, \quad (2.18)$$

where $a$ is a constant.

Solution: It isn’t convenient to make the u-substitution $u = a^2 - x^2$ because the new element of integration $du = \frac{dh}{dx} \, dx = -2x \, dx$ doesn’t appear in the integrand. Instead, we try to find a change of variable of integration that will eliminate the square root from the integrand. There is a class of transformations known as trigonometric substitutions[ABD05, Sec. 8.4] that can be useful for eliminating such square roots. For this square root, the change of variable of integration $x = a \sin(\theta), \, dx = a \cos(\theta) \, d\theta$ is useful provided that $|x| \leq a$ for all $x$ in the region of integration. This transformation eliminates the square root because of the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$. We have

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2(\theta)} = a \cos(\theta). \quad (2.19)$$

\(^1\)If the equation $x = h(u)$ is solved for $u$ as a function of $x$, the resulting equation $u = u(x)$ is called the inverse of $x = h(x)$.\/
To determine the limits of integration in terms of the new variable of integration, we need to invert the equation for $x$ in terms of $\theta$ to find the equation for $\theta$ in terms of $x$. The result is $\theta(x) = \arcsin \frac{x}{a}$, which yields $\theta(-a) = -\frac{\pi}{2}$ for the lower limit and $\theta(a) = \frac{\pi}{2}$ for the upper limit.

Consequently, the transformed integral becomes

$$I = a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) \, d\theta.$$  \hspace{1cm} (2.20)

Finally, we apply the half-angle formula $\cos^2(\frac{\theta}{2}) = \frac{1 + \cos(\theta)}{2}$ to obtain

$$I = a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2} \, d\theta$$

$$= \frac{a^2}{2} \left[ \theta + \frac{1}{2} \sin(2\theta) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{a^2 \pi}{2}.$$  \hspace{1cm} (2.21)

We will need to apply trigonometric substitutions from time to time throughout the course. For an example applied to an electrical problem, see Sample Problem 3.3 in Sec. 3.6.5.
Chapter 3

Charge Integrals

3.1 Introduction

Charge integrals are encountered almost at the beginning of the electricity and magnetism course, because they are needed to determine the electric field that is produced by electric charge which is distributed over a region of space. They are needed, for example, if we want to calculate the electric field produced at a point in space by an electrically charged wire, disk, cylindrical surface, or solid cube, even if the charge is uniformly distributed throughout the object. Since the functions in the integrands of the resulting integrals vary with position as the element of integration moves throughout the charged object, we will be integrating functions of more than one variable. Consequently, the chapter begins with a brief introduction to multivariable functions. That section is then followed by a short discussion of mass integrals. Although mass integrals do not appear in the theory of electricity and magnetism, they are described here because, mathematically, they are the same as charge integrals with charge replaced by mass, and they are easier to visualize physically for most students. Moreover, many students have already had some experience with mass integrals in a previous course in mechanics, and it can be helpful to them to relate what they know to the topic at hand.

After the introductory material mentioned above, the definition of charge integrals is provided, followed by a discussion about how to convert the charge integrals into space integrals, such as line and surface integrals, (which are treated in the following chapters). The chapter concludes with a thorough treatment of the application of charge integrals in the field of electrostatics.

3.2 Multivariable Functions

In our discussion of definite integrals in Chapter 2, the integrand $f(x)$ we dealt with was a function of the single variable $x$. Since the integrand in charge integrals can be a function of the position $(x, y, z)$ of the element of integration,
it is a function of the 3 variables \(x, y,\) and \(z\). We indicate this by writing the function as \(f(x, y, z)\) and saying that \(f\) is a function of \(x, y,\) and \(z\). Examples of such functions include \(f(x, y, z) = x^2 + yz\) and \(f(x, y, z) = x \sin(ky)\). Note that writing \(f(x, y, z)\) does not preclude the possibility that the function is independent of one or more of the variables any more than writing \(f(x)\) precludes the possibility that \(f(x)\) is a constant. The values of \(x, y,\) and \(z\) for which \(f(x, y, z)\) is defined make up what is called the “domain” of \(f(x, y, z)\).

Functions can be functions of any number of variables, and the arguments of the function are written accordingly. For example, if \(f\) is a function of both position in space and time, we can write \(f(x, y, z, t)\). Or if we know that \(f\) doesn’t vary with \(z\) or \(t\), we can write \(f(x, y)\).

### 3.3 Mass Integrals

Charge integrals are very much like mass integrals, which you probably encountered in your mechanics course. They refer to the integration of some integrand over the distribution of mass in an extended object. They are written as

\[
I = \int_M f(x, y, z) \, dM. \tag{3.1}
\]

The symbol \(M\) is used for two purposes. First, it is used to label the entire mass distribution, and second, it is used to indicate the total mass of the object quantitatively. Consequently, it makes sense to say that “the total mass of \(M\) is \(3\) kg”\(^{-1}\). The symbol \(M\) at the bottom of the integral sign in the above mass integral indicates that we are to integrate over the extended mass \(M\).

Recall, for example, that value of \(x\) for the center of mass of a solid object of mass \(M\) is given by the mass integral[HRW08, Sec. 9-2]

\[
x_{\text{com}} = \frac{1}{M} \int_M x \, dM, \tag{3.2}
\]

where the \(x\) in the integrand is the value of \(x\) at the location of the element of integration \(dM\). The values of \(y\) and \(z\) at the center of mass are given by the same integral with \(x\) replaced in the integrand by \(y\) and \(z\) respectively. Recall also, that the rotational inertia of a solid object of mass \(M\) about a given axis of rotation is given by[HRW08, Sec. 10-7]

\[
I_{\text{rot}} = \int_M r^2 \, dM, \tag{3.3}
\]

where \(r\) is the perpendicular distance of the element of integration \(dM\) from the rotational axis.

Mass integrals are defined exactly the same as are charge integrals in Section 3.4 but with charge \(Q\) replaced by mass \(M\). Consequently, mass integrals can serve as a convenient analogy for charge integrals for those students who find it easier to visualize mass integrals than charge integrals.
3.4 Definition

Charge integrals refer to the integration of some integrand over the distribution of charge in an extended body. Similar to the symbol \( M \), the symbol \( Q \) is used for two purposes. First, it is used to label the entire extended body of charge and secondly, it is used to indicate the total amount of charge quantitatively that the body carries. For example, let \( Q \) represent a uniformly charged sphere with total charge of 5 coulombs. Then, the total charge of the charge distribution \( Q \) is \( Q = 5 \, C \).

The charge integral of an integrand \( f(x,y,z) \) over a charge distribution \( Q \) is written

\[
I = \int_Q f(x, y, z) \, dQ. \tag{3.4}
\]

The \( Q \) is placed on the integration sign to indicate that the integral is over the entire charge distribution which is labeled as \( Q \).

The charge integral is defined by the same procedure used to define the definite integral in Section 2.2 with one change. Instead of considering the interval from \( a \) to \( b \) along the \( x \) axis and partitioning it into tiny intervals \( \Delta x_i \), we consider the charge distribution \( Q \) and partition it into tiny chunks of charge \( \Delta Q_i \).

The complete procedure is as follows.

1. Break the charge distribution \( Q \) into \( n \) tiny chunks of charge. The amount of charge in the \( i^{th} \) chunk is \( \Delta Q_i \), with \( i \) ranging from 1 to \( n \). We do the partition so that all of the tiny chunks of charge tend towards zero simultaneously as the number \( n \) of the chunks increases towards infinity.

2. Evaluate the integrand at the location \( (x_i, y_i, z_i) \) of each chunk of charge and multiply it times the amount of charge in that chunk.

3. Add together all \( n \) of the above products of the multiplications.

4. Take the limit of the sum as the number \( n \) of tiny intervals tends to infinity.

The result of this process is the charge integral expressed in Eqn. (3.4). After doing the first step described above, all of the remaining steps can be expressed mathematically by a single equation

\[
\int_Q f(x, y, z) \, dQ = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta Q_i. \tag{3.5}
\]

Equation (3.5), taken together with the subdivision of \( Q \) into the \( n \) tiny chunks of charge \( \Delta Q_i \), gives the mathematical definition of the charge integral. An example of the application of this definition is worked out in detail in Section 3.6.4 Coulomb’s Law for Distributed Charges.
3.5 Conversion Into Space Integrals

In the review of definite integrals given in Chapter 2, it was pointed out that the definition of the definite integral is not very useful for evaluating the integral. The same is true for charge integrals. We need to find another way to evaluate charge integrals. The first step in that direction is to convert the charge integral into one that integrates over the region of space occupied by the charge distribution. Such integrals can be referred to as space integrals. The conversion is accomplished by employing the concept of “charge density”.

If the charge \( Q \) is distributed along a straight or curved line segment, the relevant charge density is the linear charge density \( \lambda(s) \), which is charge per unit length along the line. Since the density can depend on position along the line, it is indicated that \( \lambda \) can be a function of \( s \), which is the arc length along the line from the beginning of the segment to the point of interest where the linear charge density is to be specified. The linear charge density is defined at a position of arclength \( s_i \) along the line by

\[
\lambda(s_i) = \frac{\Delta Q_i}{\Delta s_i},
\]

where \( \Delta Q_i \) is a tiny chunk of charge with length \( \Delta s_i \) on the line located at arc length \( s_i \).

When the charge distribution on the line segment is partitioned into tiny chunks of charge \( \Delta Q_i \), it is simultaneously partitioned into tiny lengths \( \Delta s_i \) as well. The two are related by

\[
\Delta Q_i = \lambda(s_i) \Delta s_i. \tag{3.7}
\]

This can be used to convert a charge integral along a curve \( C \) into a space integral over \( C \) by simply substituting Eqn. (3.7) into the definition of the charge integral given in Eqn.(3.5) to obtain

\[
\int_Q f(x, y, z) \, dQ = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \lambda(s_i) \Delta s_i. \tag{3.8}
\]

The right-hand side of Eqn. (3.8) clearly expresses a new integration process, one that integrates spatially over the curve \( C \). The result is called a line integral, which is defined by

\[
\int_C f(x, y, z) \, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta s_i. \tag{3.9}
\]

The \( C \) at the bottom of the integral sign indicates that the integration is over the curve \( C \). As always, the integrand \( f(x, y, z) \) is to be evaluated at the location of the element of integration \( ds \). This means that it is evaluated only at points on the curve \( C \) during the integration process.
3.5. CONVERSION INTO SPACE INTEGRALS

Using this definition of the line integral, the expression for the charge integral in Eqn. (3.8) can be written

\[ \int_Q f(x, y, z) \, dQ = \int_C f(x, y, z) \lambda(s) \, ds. \]  (3.10)

This is the desired conversion of the charge integral into a space integral in the case when the charge is distributed along a line. The conversion can be made more directly in the future by simply making the change of variable of integration

\[ dQ = \lambda(s) \, ds \]  (3.11)

with the appropriate change in the domain of the integration.

Line integrals are important in their own right, apart from their connection to charge integrals. Indeed, you may have already encountered them in your mechanics class when you studied the work done on a body as it moves along a curved path under the influence of a force. Line integrals appear in the integral form of Faraday’s law and the Ampere-Maxwell law in the theory of electricity and magnetism. Chapter 4 Line Integrals treats these integrals in detail.

If the charge \( Q \) is distributed on a flat or curved surface \( S \) (either opened or closed), the relevant charge density is the surface charge density \( \sigma \), which is charge per unit area along the surface. Although it isn’t indicated here with symbols, \( \sigma \) can vary with position on the surface. This time, we convert the charge integral into a space integral over the surface \( S \) by making the change of variable of integration

\[ dQ = \sigma \, dS, \]  (3.12)

where \( dS \) is an element of surface area located at an arbitrary point on the surface and \( \sigma \) is evaluated at that point. The result is

\[ \int_Q f(x, y, z) \, dQ = \int_S f(x, y, z) \sigma \, dS, \]  (3.13)

where the right-hand side represents a surface integral over the surface \( S \).

Surface integrals are important in their own right, apart from their connection to charge integrals. Surface integrals appear in the integral form of Gauss’ law for electricity, Gauss’ law for magnetism, Faraday’s law, and the Ampere-Maxwell law in the theory of electricity and magnetism. They are treated for flat surfaces in Chapter 5 Double Integrals and for general surfaces in Chapter 6 Surface Integrals.

If the charge \( Q \) is distributed in a three-dimensional volume \( V \), the relevant charge density is the volume charge density \( \rho(x, y, z) \), which is charge per unit volume. This time, we convert the charge integral into a space integral over the volume \( V \) by making the change of variable of integration

\[ dQ = \rho(x, y, z) \, dV, \]  (3.14)
where $dV$ is an element of volume located at an arbitrary point in the volume $V$ and $\rho(x, y, z)$ is evaluated at that point. The result is

$$\int_Q f(x, y, z) \, dQ = \int_V f(x, y, z) \rho(x, y, z) \, dV,$$

where the right-hand side represents a volume integral over the volume $V$.

Volume integrals (also known as triple integrals) are not as deeply imbedded in the theory of electricity and magnetism as are line and surface integrals, but they are still needed occasionally. Their primary application is for using Coulomb’s law for volume distributions of charge and for determining the total amount of charge in volumes with known charge density in the application of Gauss’ law for electricity. A brief summary of these integrals is given in Appendix A.

### 3.6 Application to Coulomb’s Law

#### 3.6.1 Introduction

In this section, we set up the mathematics for determining the force $\vec{F}_q$ produced on a point charge $q$ by charge $Q$ which is distributed throughout a finite region of space. That region can be a straight or curved line, or a flat or curved surface. We begin by putting Coulomb’s law into vector form so that it includes both the magnitude and direction of the force $\vec{F}_q$ due to a point charge. Next, we obtain expressions for the $x$, $y$, and $z$ components of that force. After a short discussion about how these expressions can be used to find the net force produced by a collection of point charges, we extend the treatment to include the force $\vec{F}_q$ due to a continuous distribution of charge over a region of space by applying charge integration. Then, we learn how to convert the charge integrals into space integrals including line, surface, and double integrals depending on the geometry of the distribution of charge. The section concludes by determining the force $\vec{F}_q$ due to charge distributed along a straight line segment. The resulting line integral is finally expressed as a definite integral.

#### 3.6.2 The Vector Form of Coulomb’s Law

In most fundamental-physics texts covering electricity and magnetism, the magnitude of the electrostatic forces between two point charges is given by a formula, whereas the directions of the forces are stated in words [the force on one charge is directly away from the other one (the charges repel each other) if the charges have the same sign or is directly towards the other one (the charges attract each other) if the charges have opposite signs]. For our purposes here, however, it is more convenient to express the force on one of the charges as a vector in a single vector equation which gives both the magnitude and direction of the force. We do that in this section.
Consider the force \( \vec{F}_q \) on a point charge \( q \) caused by another point charge \( Q \) that is a distance \( r \) away. According to Coulomb’s law\([HRW08, \text{Eqn. (21.1)}]\), the magnitude of \( \vec{F}_q \) is

\[
F_q = k \frac{|q||Q|}{r^2},
\]

(3.16)

where \( k \) is a constant known as the electrostatic constant. Hence, we can express the vector \( \vec{F}_q \) by a single vector equation

\[
\vec{F}_q = k \frac{|q||Q|}{r^2} \hat{l},
\]

(3.17)

if we can identify a unit vector \( \hat{l} \) that points away from \( Q \) when \( q \) and \( Q \) have the same sign (they are either both positive or both negative) and points toward \( Q \) when \( q \) and \( Q \) have opposite sign (one is positive and the other is negative). This can be done by defining the vector \( \vec{r} \) to be the vector from \( Q \) to \( q \). Then, \( \vec{r} \) has magnitude \( r \) and direction pointing away from \( Q \). We can then use \( \vec{r} \) to define our unit vector \( \hat{l} \) as

\[
\hat{l} = \frac{qQ}{|q||Q|} \frac{\vec{r}}{r}.
\]

(3.18)

This works because its magnitude is clearly one and its direction is in the same direction as \( \vec{r} \) (away from \( Q \)) when \( q \) and \( Q \) have the same sign (since the product \( qQ \) is positive) and in the opposite direction as \( \vec{r} \) (towards \( Q \)) when \( q \) and \( Q \) have the opposite sign (since \( qQ \) is negative). Substitution of Eqn. (3.18) into Eqn. (3.17) gives the desired vector equation for the force

\[
\vec{F}_q = k \frac{qQ}{r^3} \vec{r}.
\]

(3.19)

This equation is very convenient to use when we wish to express the force in vector form. When you use it, you must keep in mind that this is the force that \( q \) feels because of the presence of \( Q \) and is not the force on \( Q \) due to \( q \). You must remember, also, that the vector \( \vec{r} \) points away from \( Q \). (If you need an expression for the force on \( Q \) caused by the presence of \( q \), you can use the same formula but then you take the vector \( \vec{r} \) to point from \( q \) to \( Q \) instead of from \( Q \) to \( q \).) Don’t let the form of Eqn. (3.6.2) mislead you into thinking that the force follows an inverse cube law. Since the magnitude of the vector \( \vec{r} \) in the numerator is \( r \), the force still follows the inverse square law as specified by the original form of Coulomb’s law.

The Vector Between Two Points

Throughout the electricity and magnetism course, we are going to repeatedly make use of a vector that has one end at a given point in space and the other end at another given point. Indeed, we have already encountered one, the vector \( \vec{r} \) in the vector form of Coulomb’s law. It is important that you be able to express such a vector mathematically in terms of the coordinates of the two points and
be able to determine its scalar components. Consequently, we pause here to obtain the required relationships.

First, note that the actual location of the vector is of no importance. All vectors with the same length and direction are all equivalent. Likewise, all vectors with the same scalar components are equivalent. Consequently, all we need to determine is the scalar components of the vector.

Consider a vector \( \vec{r} \) that extends from the point \( P_1 \) located at \((x_1, y_1, z_1)\) to the point \( P_2 \) located at \((x_2, y_2, z_2)\) as shown in Figure 3.1. To help us to visualize the vector in three dimensions, the figure includes two vertical dashed lines which project the end points of the vector down to the plane \( z = 0 \) and four horizontal dashed lines in that plane which show the \( x \) and \( y \) coordinates of those projection points.\(^1\) The horizontal vector \( \vec{r}_p \) is the projection of \( \vec{r} \) onto the \( z = 0 \) plane. To determine a mathematical expression for the vector, it is convenient to shift it so that its tail is at the origin without changing its length or direction as shown in Fig. 3.2. This is done by moving each point located on the vector at \((x, y, z)\) to the new location \((x - x_1, y - y_1, z - z_1)\). This relocates \( P_1 \) to \((0, 0, 0)\) and \( P_2 \) to \((x_2 - x_1, y_2 - y_1, z_2 - z_1)\). As mentioned above, this relocation of the vector does not change it in any way. Again, the vertical dashed line shows the projection of \( P_2 \) onto the \( z = 0 \) plane and the two horizontal dashed lines in the \( z = 0 \) plane show the \( x \) and \( y \) coordinates of that projection. The projection of \( \vec{r} \) onto the \( z = 0 \) plane is again shown as the horizontal vector \( \vec{r}_p \). In addition, the figure includes the three vectors \((x_2 - x_1)\hat{i}, (y_2 - y_1)\hat{j}, \) and \((z_2 - z_1)\hat{k}\) in positions that make it obvious that \( \vec{r}_p \) is the vector sum of the two vectors \((x_2 - x_1)\hat{i}\) and \((y_2 - y_1)\hat{j}\) and that \( \vec{r} \) is the vector sum of the two vectors \( \vec{r}_p \) and \((z_2 - z_1)\hat{k}\). The combination of these two

\(^1\)The concept of projection is discussed in detail in Sec. 6.4.2.
results gives the mathematical expression we are searching for

$$\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}. \quad (3.20)$$

This expression immediately reveals that the x, y, and z components of the vector are

$$r_x = x_2 - x_1, \quad (3.21)$$
$$r_y = y_2 - y_1, \quad (3.22)$$
$$r_z = z_2 - z_1, \quad (3.23)$$

and the magnitude of the vector is

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \quad (3.24)$$

Another interesting result revealed by the above expression for the vector $\vec{r}$ is that it can be expressed by the difference between two vectors as

$$\vec{r} = \vec{r}_2 - \vec{r}_1, \quad (3.25)$$

where $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ is the vector from the origin to the point $P_1$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ is the vector from the origin to the point $P_2$

This ends our pause to discuss the vector between two specified points. Now, we incorporate what we learned into the discussion about the vector form of Coulomb’s law. Let the charge $q$ be located at the point $(x_p, y_p, z_p)$ and the charge $Q$ be located at the point $(x, y, z)$. Then, according to Eqn. (3.20), the vector $\vec{r}$ from $Q$ to $q$ is given by

$$\vec{r} = (x_p - x)\hat{i} + (y_p - y)\hat{j} + (z_p - z)\hat{k}. \quad (3.26)$$

Substitution of this expression into the vector form of Coulomb’s law

$$\vec{F}_q = k \frac{qQ}{r^3} \vec{r}$$

gives us the vector form of Coulomb’s law in terms of the coordinates of the two charges. The result is

$$\vec{F}_q = k \frac{qQ}{r^3} [(x_p - x)\hat{i} + (y_p - y)\hat{j} + (z_p - z)\hat{k}], \quad (3.27)$$

where $r$ is given by

$$r = \sqrt{(x_p - x)^2 + (y_p - y)^2 + (z_p - z)^2}. \quad (3.28)$$

---

2The notation for the locations of the two charges was chosen with our future use of these equations in mind. The point $(x_p, y_p, z_p)$ will usually be a fixed point of interest, whereas the point $(x, y, z)$ will usually be a variable location.
It is easy to pick off the scalar components of the vector $\vec{F}_q$ from Eqn. (3.27). The results are:

\[(F_q)_x = k \frac{qQ}{r^3}(x_p - x),\]  
\[(F_q)_y = k \frac{qQ}{r^3}(y_p - y),\]  
\[(F_q)_z = k \frac{qQ}{r^3}(z_p - z).\] 

These results are in the optimum form to use when we need to add together the forces produced by a collection of point charges. We will also use these results to evaluate the space integrals that arise when we integrate Coulomb’s law over charge distributions.

CHECKPOINT: Apply Eq. (3.29) to determine under what conditions the $x$ component of the force on $q$ due to $Q$ will be negative. Then check to see if your conditions are correct by using conceptual reasoning (without mathematics) based on the fact that like charges repel each other and unlike charges attract each other.

Sample Problem 3.1 Consider a point charge $Q$ with charge $-7$ C and mass 2 kg and another point charge $q$ with charge 4 C and mass 8 kg, which interact with each other in free space (i.e. the only forces acting on the particles are the forces they exert on each other). Assume that the only forces the particles are exerting on each other are the electrical force and the force of gravity. Determine a symbolic expression and a numerical expression for the acceleration $\vec{a}_Q$ of $Q$ at the instant when $Q$ is located at $(3 \text{ m}, -6 \text{ m}, 5 \text{ m})$ and $q$ is located at $(2 \text{ m}, 4 \text{ m}, 3 \text{ m})$, where the coordinates of each point are indicated in meters using the notation $(x,y,z)$.

Solution: The presentation of the solution to this problem follows the standard format that is used in the author’s classes. Since a symbolic expression is required, we need to assign symbols to all of the given quantities. We do so in a paragraph at the beginning, which states the symbols and the numerical values with units for each given quantity.

Givens: $Q = -7$ C, $m_Q = 2$ kg, $x_Q = 3$ m, $y_Q = -6$ m, $z_Q = 5$ m, $q = 4$ C, $m_q = 8$ kg, $x_q = 2$ m, $y_q = 4$ m, $z_q = 3$ m.

The symbolic solution should contain only the symbols for these quantities and those for known physical quantities such as $k$ for the electrostatic constant and $G$ for the gravitational constant.

According to Newton’s Second Law[HRW08, Sec. 5-6], the acceleration of $Q$ is given by

$$a_Q = \frac{\vec{F}_Q}{m_Q},$$

where the net force $\vec{F}_Q$ on $Q$ is the sum of the electric force and the gravitational force exerted by $q$. Since that net force is equal to minus the net force $\vec{F}_q$ on $q$
exerted by $Q$ according to Newton’s third law [HRW08, Sec. 5-8], we have

$$a_Q = -\frac{F_q}{m_Q}.$$

Because the force $F_q$ is a vector, we need to use the vector form of Coulomb’s law for the electrical force as given in Eqn. (3.6.2)

$$(F_q)_{\text{electrical}} = k \frac{qQ}{r^3} \hat{r},$$

where, in this case the vector $\vec{r}$ is given by

$$(x_q - x_Q)i + (y_q - y_Q)j + (z_q - z_Q)k,$$

and its magnitude is given by

$$r = \sqrt{(x_q - x_Q)^2 + (y_q - y_Q)^2 + (z_q - z_Q)^2}.$$

We need, also, a corresponding vector form of Newton’s universal law of gravitation [HRW08, Sec. 13-2] for the gravitational force. The gravitational and electrical laws are so similar to each other, that the vector form of the gravitational law can be obtained by making a slight modification of the vector form of the electrical law. Note that the magnitude of the electrical force is

$$(F_q)_{\text{electrical}} = k \frac{|q||Q|}{r^2},$$

whereas the magnitude of the gravitational force is [HRW08, Sec. 13.2]

$$(F_q)_{\text{gravitational}} = G \frac{m_q m_Q}{r^2},$$

where $G$ is the gravitational constant. The electrical force on $q$ is away from $Q$ (in the same direction as $\hat{r}$) if the product $qQ$ is positive and is towards $Q$ (in the opposite direction as $\hat{r}$) if the same product is negative, whereas the gravitational force is always towards $Q$ (always in the opposite direction as $\hat{r}$) and the product $m_q m_Q$ is always positive. These observations show that the vector form of Coulomb’s law given above can be modified to obtain the vector form of Newton’s universal law of gravitation as

$$(F_q)_{\text{gravitational}} = -G \frac{m_q m_Q}{r^3} \hat{r}.$$

The net force on $q$ is the vector sum of the electrical and gravitational forces, leading to an expression for the acceleration of $Q$ of

$$a_Q = -\frac{(kqQ - G m_q m_Q)}{m_Q} \frac{\hat{r}}{r^3}.$$

---

3See Eqn. (3.26).
4See Eqn. (3.28).
At this point, it may appear that we have our final symbolic expression. Although $\vec{r}$ and $r$ are not given quantities, they are expressed in terms of given quantities in the equations just above. The standard format used in the author’s classes for analytic homework and exams, however, does not permit this. It requires that right-hand-side of the symbolic expression involve only symbols for the given quantities and physical constants of nature, with no intermediary quantities that are not givens, even if equations have been given that relate these quantities to the given quantities. To use the above equation to obtain a symbolic solution in the standard format, we must substitute in the equations for $\vec{r}$ and $r$ in terms of the givens. The result is

$$a_Q = -\frac{(kqQ - Gm_qm_Q)(x_q - x_Q)\hat{i} + (y_q - y_Q)\hat{j} + (z_q - z_Q)\hat{k}}{m_Q[(x_q - x_Q)^2 + (y_q - y_Q)^2 + (z_q - z_Q)^2]^{3/2}}.$$  

This is our symbolic solution. If this were an analytic homework or exam problem, we would need to put a box around the right-hand side of this equation.

To obtain the numerical solution, the format calls for you to write the same equation with the symbols for the givens replaced by their numerical values. This makes it easy to check to make sure you have used the correct numerical values. Then, you do all of the arithmetic and calculate the numerical answer. The result, in this case, would be

$$a_Q = -\frac{[(8.99 \times 10^9)(-7)(4) - (6.67 \times 10^{-11})(8)(2)][(2 - 3)\hat{i} + (4 + 6)\hat{j} + (3 - 5)\hat{k}]}{2[(2 - 3)^2 + (4 + 6)^2 + (3 - 5)^2]^{3/2}}$$

$$= (-1.17\hat{i} + 11.7\hat{j} - 2.34\hat{k}) \times 10^8 \text{ms}^{-2}.$$  

Again, we would put a box around the extreme right-hand side of this equation if this were an analytic homework or exam problem.\(^5\)

### 3.6.3 Coulomb’s Law For a Collection of Point Charges

Suppose that the charge $q$ is in the presence of $n$ other point charges $Q_i$ where $i$ is an integer ranging from 1 to $n$. Then, according to the principle of superposition for electrical forces\([HRW08, \text{Eqn. 21-7}],\) the net force $\vec{F}$ on $q$ due to these charges is just equal to the vector sum of all of the forces $\vec{F}_i$ on $q$ due to each of the other charges

$$\vec{F} = \sum_{i=1}^{n} \vec{F}_i.$$  

\(^5\)It is worthwhile to note that, if we had calculated the two forces separately, we would have found that the gravitational force is much, much smaller than the electrical force, and it can be completely neglected in this calculation. This is the typical situation when the masses involved are not on the planetary scale.
Making use of the vector form of Coulomb’s law for each of the forces \( \vec{F}_i \), we can rewrite the sum of the forces as

\[
\vec{F} = \sum_{i=1}^{n} k \frac{qQ_i}{r_i^3} \vec{r}_i,
\]

where \( \vec{r}_i \) is the vector from \( Q_i \) to \( q \).

Although this equation is a simple way to write what must be done in order to evaluate the net force, it represents a sum of several vectors, which can be complicated to evaluate. As always, the easiest and most accurate way to evaluate a sum of vectors is to find the components of each of the vectors and add all of the \( x \) components together to get the \( x \) component of the net force and similarly for the \( y \) and \( z \) components. This is straightforward to do because the components of the force on \( q \) due to each charge \( Q_i \) can be found directly by using Eqns. (3.29) – (3.31). The results are

\[
(F_q)_x = k \sum_{i=1}^{n} \frac{qQ_i}{r_i^3} (x_p - x_i),
\]

(3.34)

\[
(F_q)_y = k \sum_{i=1}^{n} \frac{qQ_i}{r_i^3} (y_p - y_i),
\]

(3.35)

\[
(F_q)_z = k \sum_{i=1}^{n} \frac{qQ_i}{r_i^3} (z_p - z_i),
\]

(3.36)

where \((x_i, y_i, z_i)\) is the location of the charge \( Q_i \).

**Sample Problem 3.2** Determine the magnitude \( F_q \) of the net force exerted on the charge \( q \) located at \( z = z_p \) on the \( z \) axis by a charge \( q_1 \) located at \( x = x_1 \) on the \( x \) axis and a charge \( q_2 \) located at \( y = y_2 \) on the \( y \) axis.

**Solution:** Since we are not given numerical values for most of the given quantities, we list only their symbols in the list of givens, and we obtain only a symbolic answer. Since the charges are located on the cartesian axes, we know that some of the coordinates of the charges are zero.

**Givens:** \( q, x_p = 0, y_p = 0, z_p, Q_1, x_1, y_1 = 0, z_1 = 0, Q_2, x_2 = 0, y_2, z_2 = 0 \).

According to the standard format for analytic homework and exam problems, we do not substitute any numerical values for their symbols until after we have obtained the symbolic answer and are beginning to compute the numerical answer. There is one exception to that rule, however, which is that we do substitute the numerical value in for the symbol right at the beginning if the numerical value is zero. Consequently, in this example, we set \( x_p, y_p, y_1, z_1, x_2, \) and \( z_2 \) equal to zero throughout. When this is done in Eqns. (3.34) – (3.36), we obtain

\[
(F_q)_x = k \sum_{i=1}^{n} \frac{qQ_i}{r_i^3} (x_p - x_i) = kq \frac{q_1}{r_1^3} (-x_1),
\]
\[ (F_q)_y = k \sum_{i=1}^{n} \frac{qQ_i}{r_1^3} (y_p - y_i) = k \frac{q_2}{r_2^3} (-y_2), \]

\[ (F_q)_z = k \sum_{i=1}^{n} \frac{qQ_i}{r_1^3} (z_p - z_i) = kq \left[ \frac{q_1}{r_1^3} + \frac{q_2}{r_2^3} \right] z_p, \]

where
\[ r_1 = \sqrt{(x_p - x_1)^2 + (y_p - y_1)^2 + (z_p - z_1)^2} = \sqrt{x_1^2 + z_p^2}, \]

\[ r_2 = \sqrt{(x_p - x_2)^2 + (y_p - y_2)^2 + (z_p - z_2)^2} = \sqrt{y_2^2 + z_p^2}. \]

Since the magnitude of \( \vec{F}_q \) is given by
\[ F_q = \sqrt{(F_q)_y^2 + (F_q)_z^2}, \]

we have for the final symbolic answer
\[ F_q = kq \sqrt{\frac{q_1^2 y_1^2}{(x_1^2 + z_p^2)^3} + \frac{q_2^2 y_2^2}{(y_2^2 + z_p^2)^3} + \left[ \frac{q_1}{\sqrt{x_1^2 + z_p^2}} \right]^2 + \left[ \frac{q_2}{\sqrt{y_2^2 + z_p^2}} \right]^2} \frac{z_p^2}{z_p}. \]

3.6.4 Coulomb’s Law For Distributed Charges

Suppose now that the point charge \( q \) located at the point \((x_p, y_p, z_p)\) is in the presence of a continuous distribution of charge, such as some charge spread out over a curved wire or a plane surface or the interior of a solid ball. Coulomb’s law can not be applied directly to obtain the electrical force exerted on \( q \) by the charge distribution. Instead, we mentally break up the charge distribution into many tiny chunks of charge, each of which is small enough that we can consider it to be a point charge and then apply Coulomb’s law. Next, we sum the forces due to all of the chunks, just as we did for a collection of point charges in the previous section. The resulting sum is a good approximation of the net force on \( q \). To make the approximation an exact result for the net force, we take the limit as the size of the chunks tend to zero and the number of chunks tend to infinity.

The complete procedure is as follows.

1. Break the charge distribution \( Q \) into \( n \) tiny chunks of charge. The amount of charge in the \( i^{th} \) chunk is \( \Delta Q_i \) with \( i \) ranging from 1 to \( n \). We do the partition so that all of the tiny chunks of charge tend towards zero simultaneously as the number \( n \) of the chunks increases towards infinity.

2. Apply the vector form of Coulomb’s law to obtain the force exerted on \( q \) by each chunk as if it were a point charge. For the \( i^{th} \) chunk, the force is
\[ \vec{F}_i = kq \frac{\vec{r}_i}{r_i^2} \Delta Q_i. \]
3. Add together all n of these forces.

4. Take the limit of the sum as the number n of tiny intervals tends to infinity.

After doing the first step described above, all of the remaining steps can be expressed mathematically by a single equation

\[ \vec{F}_q = \lim_{n \to \infty} \sum_{i=1}^{n} kq \frac{\vec{r}_i}{r_i^3} \Delta Q_i \]  

(3.38)

Comparison of this procedure with the definition of charge integrals given in Sec. 3.4 shows that the right-hand side of the above equation is just a charge integral of the vector integrand

\[ kq \frac{\vec{r}}{r^3}. \]

Consequently, the net force on q due to the charge distribution Q can be written

\[ \vec{F}_q = kq \int_{Q} \frac{\vec{r}}{r^3} dQ, \]  

(3.39)

where \( \vec{r} \) is the vector from the element of integration \( dQ \) to the point charge \( q \).

Although this equation is a simple way to write what must be done in order to evaluate the net force, it represents a sum of vectors, which can be complicated to evaluate. As always, the easiest and most accurate way to evaluate a sum of vectors is to find the components of each of the vectors and add all of the \( x \) components together to get the \( x \) component of the net force and similarly for the \( y \) and \( z \) components. This is straightforward to do because the components of the force on \( q \) due to the element of integration \( dQ_i \) can be found directly by using Eqns. (3.29) – (3.31). The results are

\[ (F_q)_x = kq \int_{Q} \frac{x_p - x}{r^3} dQ, \]  

(3.40)

\[ (F_q)_y = kq \int_{Q} \frac{y_p - y}{r^3} dQ, \]  

(3.41)

\[ (F_q)_z = kq \int_{Q} \frac{z_p - z}{r^3} dQ, \]  

(3.42)

where the element of integration \( dQ \) is located at the point \( (x, y, z) \) and the distance \( r \) is given by Eqn (3.28)

\[ r = \sqrt{(x_p - x)^2 + (y_p - y)^2 + (z_p - z)^2}. \]

These are the main formulas that we use to evaluate the force exerted on a point charge by a distribution of charge. They are used with slight modification to determine the electric field produced at a point by the distribution of charge as well. Note, that we need to evaluate three similar but different integrals to
determine the entire vector. Each of these integrals has a scalar integrand and can be written in the generic form

$$I = \int_Q f(x, y, z) \, dQ,$$

where the integrand $f(x, y, z)$ is different for each component.

As explained in Sec. 3.5, we usually evaluate such charge integrals by converting them into space integrals (integrals over the region of space occupied by the charge distribution). In particular, if the charge is distributed along a curve $C$ with a linear charge density $\lambda(s)$, then the charge integral can be converted into a line integral of the form [Eqn. (3.10)]

$$\int_Q f(x, y, z) \, dQ = \int_C f(x, y, z) \lambda(s) \, ds.$$

Similarly, if the charge is distributed over a surface $S$ with a surface charge density $\sigma(x, y, z)$, then the charge integral can be converted into surface integral of the form [Eqn. (3.13)]

$$\int_Q f(x, y, z) \, dQ = \int_S f(x, y, z) \sigma \, dS.$$

We have not yet learned how to evaluate these space integrals; they are treated in detail in the next two chapters. There is one special case of line integrals, however, that we are equipped to deal with now. This special case is treated in the next section.

### 3.6.5 Charge Distributed Along a Straight Line

The simplest geometry is when the charge is distributed along a straight line. In this case, it is easy to convert the charge integral into an ordinary definite integral, the kind that we already know how to evaluate. To illustrate this, consider a charged, very narrow, straight rod that lies along the $x$ axis from $x = a$ to $x = b$ where $b > a$. Suppose the amount of charge on the rod varies along the rod in a known way specified as a function of $x$ by the linear charge density $\lambda(x)$. We want to determine the net force exerted by this charge distribution on a charge $q$ located at $(x_p, y_p, z_p)$.

Since the charge is distributed along a line, we apply line integrals to evaluate the charge integrals involved. The integrals for the scalar components of the net force are of the form

$$I = \int_Q f(x, y, z) \, dQ = \int_C f(x, y, z) \lambda(s) \, ds,$$

where the line integral is defined by [see Eqn. (3.9)]

$$\int_C f(x, y, z) \lambda(s) \, ds = \lim_{n \to \infty} \sum_{i=1}^n f(x_i, y_i, z_i) \lambda(s) \Delta s_i.$$

(3.44)
3.6. APPLICATION TO COULOMB’S LAW

In this case, the curve C is a straight line along the x axis. This means that $y = z = 0$ all along the integration path and that $\Delta s_i$ can be taken to be $\Delta x_i$. There is one subtle point here, however. The quantity $\Delta s_i$ is an element of length and is always positive, whereas $\Delta x_i$ is positive if we are moving along the x axis in the positive x direction or negative otherwise. Therefore, when we change $\Delta s_i$ to $\Delta x_i$, we must keep $\Delta x_i$ positive by integrating in the positive direction. This ensures that $\Delta Q_i = \lambda(x_i) \Delta x_i$ has the same sign as $\lambda(x_i)$. With this in mind, we can rewrite the above equation as

$$\int_C f(x,0,0) \lambda(s) \, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i,0,0) \lambda(x) \Delta x_i. \quad (3.45)$$

Now, we can recognize that the right-hand side of the above equation is just the definition of the definite integral of $f(x,0,0) \lambda(x)$ [see Eqn. (2.2)]. Since we must integrate in the positive x direction and $b$ is larger than $a$, we must integrate from $a$ to $b$. The result is

$$\int_Q f(x,0,0) \, dQ = \int_a^b f(x,0,0) \lambda(x) \, dx. \quad (3.46)$$

Consequently, we have reduced the problem to one of evaluating a definite integral. From this point on, we can apply the standard methods of integral calculus to evaluate the integral to determine the force on q.

We can now use the above result, combined with Eqns. (3.40) – (3.42), to express the three scalar components of the net force on the charge q. The results are

$$(F_q)_x = kq \int_a^b \frac{x_p - x}{r^3} \lambda(x) \, dx, \quad (3.47)$$

$$(F_q)_y = kq \int_a^b \frac{y_p}{r^3} \lambda(x) \, dx, \quad (3.48)$$

$$(F_q)_z = kq \int_a^b \frac{z_p}{r^3} \lambda(x) \, dx, \quad (3.49)$$

where

$$r = \sqrt{(x_p - x)^2 + y_p^2 + z_p^2}. \quad (3.50)$$

Using unit-vector notation, we can write the force as a vector

$$\vec{F}_q = kq \int_a^b \frac{\lambda(x)}{|(x_p - x)^2 + y_p^2 + z_p^2|^{3/2}} dx$$

$$+ kq(y_p \hat{j} + z_p \hat{k}) \int_a^b \frac{\lambda(x)}{|(x_p - x)^2 + y_p^2 + z_p^2|^{3/2}} dx. \quad (3.51)$$

It is worth noting that this result is easily extended to include any line integral along a straight-line path that is parallel to a coordinate axis. The line integral is reduced to a definite integral with respect to that coordinate with the integrand evaluated at the constant values the other two coordinates have along the line. Remember that the integration in the definite integral must always be done in the positive direction.
We now have the simplest form of the solution to our problem we can obtain without knowing the charge density as a function of position along the rod. Once we are given that function, we can do the integration of the two integrals to obtain the symbolic solution to the problem.

**Sample Problem 3.3**  Consider a very thin, uniformly charged linear rod with linear charge density \( \lambda_0 \) which lies along the x axis from \( x = a \) to \( x = b \), where \( a, b, \) and \( \lambda_0 \) are constants. Determine a symbolic expression for the force \( \vec{F}_q \) that the rod exerts on a point charge \( q \) located at \( (x_p, y_p, z_p) \).

**Solution:**

**Givens:** \( \lambda_0, a, b, q, x_p, y_p, z_p \)

Since the linear charge density is a constant, it can be taken outside of the integral signs, and Eqn. (3.51) can be rewritten as

\[
\vec{F}_q = kq\lambda_0[i I_1 + (y_p \hat{j} + z_p \hat{k}) I_2],
\]

where \( I_1 \) and \( I_2 \) represent the definite integrals

\[
I_1 = \int_a^b \frac{x_p - x}{[(x_p - x)^2 + \xi^2]^{3/2}} \, dx,
\]

\[
I_2 = \int_a^b \frac{dx}{[(x_p - x)^2 + \xi^2]^{3/2}},
\]

with

\[
\xi^2 = y_p^2 + z_p^2.
\]

One of the requirements for a symbolic answer is that all integrals (as well as all derivatives and vector operations such as scalar and vector products) must be evaluated. Consequently, we need to evaluate \( I_1 \) and \( I_2 \). Let’s begin with \( I_1 \). This integral can be evaluated using u-substitution as explained in Sec. 2.4.1. We begin by making the substitution \( u = x - x_p \) with \( du = \frac{dx}{dx} \, dx = dx \) to obtain

\[
I_1 = -\int_{a-x_p}^{b-x_p} \frac{u \, du}{(u^2 + \xi^2)^{3/2}}.
\]

Next, we make the substitution \( v = u^2 + \xi^2 \) with \( dv = \frac{du}{du} \, du = 2u \, du \) to obtain

\[
I_1 = -\frac{1}{2} \int_{(a-x_p)^2+\xi^2}^{(b-x_p)^2+\xi^2} v^{-3/2} \, dv
\]

\[
= -\frac{1}{2} \left[ \frac{v^{-1/2}}{-1/2} \right]_{(a-x_p)^2+\xi^2}^{(b-x_p)^2+\xi^2}
\]

\[
= \frac{1}{\sqrt{(b-x_p)^2+\xi^2}} - \frac{1}{\sqrt{(a-x_p)^2+\xi^2}}.
\]

(3.57)
We now turn our attention to $I_2$. We again start with the u-substitution $u = x - x_p$ with $du = dx$ to obtain

$$I_2 = \int_{a-x_p}^{b-x_p} \frac{du}{(u^2 + \xi^2)^{3/2}}. \quad (3.58)$$

Since the numerator of the integrand doesn’t include a factor of $x$ as it did for $I_1$, it isn’t convenient in this case to do a u-substitution directly. So, we attempt to eliminate the square root by making a trigonometric substitution (see Sec. 2.4.2). Since the two terms in the square root are added rather than subtracted as they were in Sample Problem 2.3, we need to use the trigonometric identity $\sec^2(\theta) = 1 + \tan^2(\theta)$ instead of the one we used in that sample problem. Therefore, we change the variable of integration to $\theta$ by setting $u = \xi \tan(\theta)$ with $du = \xi \sec^2(\theta) \, d\theta$ in the integral. The square root becomes $(u^2 + \xi^2)^{1/2} = \xi \sec(\theta)$, and the upper and lower limits of integration become respectively $\theta_b = \arctan\left(\frac{b-x_p}{\xi}\right)$ and $\theta_a = \arctan\left(\frac{a-x_p}{\xi}\right)$. Substitution of these changes into the integral gives

$$I_2 = \int_{\theta_a}^{\theta_b} \frac{\xi \sec^2(\theta)}{\xi^3 \sec^3(\theta)} \, d\theta = \frac{1}{\xi^2} \int_{\theta_a}^{\theta_b} \cos(\theta) \, d\theta = \frac{1}{\xi^2} (\sin \theta_b - \sin \theta_a). \quad (3.59)$$

It remains to express $\sin(\theta_b)$ and $\sin(\theta_a)$ in terms of the given quantities. This can be done by taking the sin of the arctan, but this leaves us with a messy symbolic answer. To obtain a more convenient form, we represent $\tan(\theta_x) = \frac{x-x_p}{\xi}$ geometrically in Fig. 3.3. It follows directly from the figure that $\sin(\theta_x) = \frac{b-x_p}{\sqrt{(b-x_p)^2 + \xi^2}}$. Consequently, $I_2$ is given by

$$I_2 = \frac{1}{\xi^2} \left[ \frac{b-x_p}{\sqrt{(b-x_p)^2 + \xi^2}} - \frac{a-x_p}{\sqrt{(a-x_p)^2 + \xi^2}} \right]. \quad (3.60)$$

To obtain the final symbolic answer for the force on the point charge $q$, we need to substitute the definition of $\xi^2$ given by Eqn. (3.55) into our expressions for $I_1$ and $I_2$ and then substitute them into Eqn. (3.52) for the force on $q$ due to the rod. The result is

$$F_q = k q \lambda \left[ \frac{1}{\sqrt{(b-x_p)^2 + y_p^2 + z_p^2}} - \frac{1}{\sqrt{(a-x_p)^2 + y_p^2 + z_p^2}} \right] + \frac{k q \lambda (y_{p} \hat{j} + z_p \hat{k})}{y_p^2 + z_p^2} \left[ \frac{b-x_p}{\sqrt{(b-x_p)^2 + y_p^2 + z_p^2}} - \frac{a-x_p}{\sqrt{(a-x_p)^2 + y_p^2 + z_p^2}} \right]. \quad (3.61)$$
CHAPTER 3. CHARGE INTEGRALS

PROBLEMS

(As always, all integrals in your solutions must be evaluated in terms of the given quantities unless it is specified otherwise.)

1. Find the value $x_{com}$ of $x$ at the center of mass of a very thin rod lying along the $x$ axis from $x = 0$ to $x = a$ where $a$ is a positive constant. The linear mass density (mass per unit length) of the rod is given by $\lambda(x) = Ax^4$, where $A$ is a positive constant. (Solution check: For $A = 2.00 \, kg/m^{-5}$ and $a = 3.00 \, m$, the numerical value is 2.50 m.)

2. Find the total amount of charge $Q$ on a very thin rod lying along the $x$ axis from $x = 0$ to $x = a$ where $a$ is a positive constant. The linear charge density (charge per unit length) of the rod is given by $\lambda(x) = A \sin K x$, where $A$ and $K$ are positive constants. The units of $K$ are radians/m. (Solution check: For $a = 2.00 \, m$, $A = 3.00 \, C/m$, and $K = \pi/5 \, radians/m$, the numerical value is 3.30 C.)

3. The four point charges listed below exert a net force $\vec{F}_q$ on a point charge $q = 10 \, C$ located on the $z$ axis at $z = 4 \, m$. Obtain numerical values in Newtons for the $x$, $y$, and $z$ components of $\vec{F}_q$ and for the magnitude of $\vec{F}_q$. No symbolic solutions are required. The four charges are: $q_1 = 6 \, C$ located on the $x$ axis at $x = 3 \, m$, $q_2 = -6 \, C$ located on the $x$ axis at $x = -3 \, m$, $q_3 = 7 \, C$ located on the $y$ axis at $y = 3 \, m$, and $q_4 = 7 \, C$ located on the $y$ axis at $y = 8 \, m$.

Figure 3.3: Geometrical representation of $\tan(\theta_x)$. 
4. Consider a very thin, straight rod that lies along the $x$ axis. Its left end is at $x = a$, and its right end is at $x = b$, where $a$ and $b$ are positive constants with $b > a$. The linear charge density of this rod is

$$\lambda(x) = Ax^3,$$

where $A$ is a positive constant. Obtain an expression for the force $\vec{F}_q$ that this rod exerts on a point charge $q$ located at the origin. (Solution check: For $a = 1.00 \text{ m}, b = 2.00 \text{ m}, A = 3.00 \text{ C m}^{-4},$ and $q = 4.00 \text{ C}$, the numerical value is $-1.62 \times 10^{11} \hat{i} \text{ N}$.)

5. Consider a very thin, straight rod that lies along the $x$ axis with its left end at the origin and its right end at $x = L$, where $L$ is a positive constant. The linear charge density of this rod is

$$\lambda(x) = Ax,$$

where $A$ is a positive constant. Obtain an expression for the $z$ component of the force that this rod exerts on a point charge $q$ located on the $z$ axis at $z = z_0$. (For $q = 1.00 \text{ C}, A = 2.00 \text{ C m}^{-2}, L = 3.00 \text{ m},$ and $z_0 = 4.00 \text{ m}$, the numerical value is $3.60 \times 10^9 \text{ N}$.)
CHAPTER 3. CHARGE INTEGRALS
Chapter 4

Line Integrals

4.1 Introduction

Line integrals are the simplest form of the space integrals described in Sec. 3.5. In this case, the relevant region of space is a line, either straight or curved. These are the integrals that are needed when we want to calculate the force exerted on a point charge by charge that is distributed along a very thin thread which can be considered to be a line. Line integrals are also required to calculate the magnetic field produced at a point in space by an electrical current flowing through a very thin wire. These integrals have much greater importance than that, however. The integral form of Faraday’s law and the Ampere-Maxwell law, both involve line integrals as well, even when there are no threads or wires present.

In this chapter, line integrals are defined, and it is shown how to evaluate them by converting them into definite integrals. Finally, some examples are given to show how line integrals can be applied to calculate the force exerted on a point charge by a linear distribution of charge when the line is curved.

4.2 Definition

Line integrals refer to the integration of some integrand $f(x,y,z)$ along a curve $C$ such as the one illustrated in Figure 4.1. The curve $C$ has a direction as indicated by the arrowhead at the end of the curve. As mentioned in Sec. 3.5, the line integral of an integrand $f(x,y,z)$ over a curve $C$ is written

$$ I = \int_C f(x,y,z) \, ds. \tag{4.1} $$

The $C$ is placed on the integration sign to indicate that the integral is over the curve which is labeled as $C$.

The line integral is defined by the same procedure as the one used to define the definite integral in Section 2.2 with one change. Instead of considering the
interval from \(a\) to \(b\) along the \(x\) axis and partitioning it into tiny sub-intervals \(\Delta x_i\), we consider the curve \(C\) and partition it into tiny sections of length \(\Delta s_i\). The complete procedure is as follows.

1. Break the curve \(C\) into \(n\) tiny sections. The length of the \(i^{th}\) section is \(\Delta s_i\) with \(i\) ranging from 1 to \(n\), and all of these lengths are taken to be positive regardless of the direction of integration (which is the direction of \(C\)). We do the partition so that all of the tiny sections exactly cover the curve \(C\), and their lengths tend towards zero simultaneously as the number \(n\) of the intervals increases towards infinity.

2. Evaluate the integrand at the location \((x_i, y_i, z_i)\) of each section and multiply it times the length of the section \(\Delta s_i\).

3. Add together all \(n\) of the above products of the multiplications.

4. Take the limit of the sum as the number \(n\) of tiny sections tends to infinity.

The result of this process is the line integral expressed in Eqn. (4.1). After doing the first step described above, all of the remaining steps can be expressed mathematically by a single equation

\[
\int_C f(x, y, z) \, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta s_i. \tag{4.2}
\]

Equation (4.2), taken together with the subdivision of \(C\) into the \(n\) tiny sections of length \(\Delta s_i\), gives the mathematical definition of the line integral.

Because the definitions of line integrals and definite integrals are so similar, they share many common properties. For example, if the integrand \(f(x, y, z)\)
4.3. LINE INTEGRAL OF A CONSTANT

includes a constant factor $K$ so that $f(x, y, z) = Kg(x, y, z)$, then we have

$$\int_C Kg(x, y, z) \, ds = K \lim_{n \to \infty} \sum_{i=1}^{n} g(x_i, y_i, z_i) \Delta s_i = K \int_C g(x, y, z) \, ds, \quad (4.3)$$

where the constant $K$ has been factored outside of the sum because it is included in every term. The result is that the constant factor in the integrand can be moved outside of the integral sign just as it can be done with definite integrals.

Two more basic properties of line integrals that are shared with definite integrals are:

$$\int_C [f(x, y, z) + g(x, y, z)] \, ds = \int_C f(x, y, z) \, ds + \int_C g(x, y, z) \, ds \quad (4.4)$$

and

$$\int_{C_1+C_2} f(x, y, z) \, ds = \int_{C_1} f(x, y, z) \, ds + \int_{C_2} f(x, y, z) \, ds. \quad (4.5)$$

These properties again follow directly from the definition of line integrals given in Eqn. (4.2).

There is, however, an important difference between the line integral and the definite integral that should be pointed out. As explained in the review of definite integrals in Chapter 2, if we reverse the direction of integration of a definite integral, we obtain the same result but with opposite sign. This results from the fact that the sub-interval $\Delta x_i$ is positive when we integrate in the positive $x$ direction, but it is negative when we integrate in the negative $x$ direction. This leads to the result

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx.$$

In the case of line integrals, however, the situation is the opposite. As long as the definition of the integrand $f(x, y, z)$ is independent of the curve $C$, reversing the direction of the curve (which reverses the direction of integration for the line integral) does not change the value of the line integral at all.\(^1\) In that case, we have

$$\int_C f(x, y, z) \, ds = \int_{C'} f(x, y, z) \, ds, \quad (4.6)$$

where $C'$ is identical to $C$ except that it has the opposite direction.

### 4.3 Line Integral of a Constant

A very important special case of line integrals to keep in mind is the one with integrand $f(x, y, z) = K$ where $K$ is a constant. Factoring the constant outside

\(^1\)In Chapter 7, we encounter integrals, called “work integrals”, which can be written as line integrals with a scalar integrand that does depend on the curve $C$. Those integrals do indeed reverse their sign when the direction of integration is reversed.
of the integral sign, we have
\[ I = \int_C K \, ds = K \int_C ds. \]

According to the defining equation for line integrals Eqn. (4.2), the line integral on the right-hand-side of the above equation is
\[ \int_C ds = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta s_i. \]

Since the sum on the right-hand-side of the above equation adds together the lengths of all of the tiny sections that make up the curve \( C \), the result of the sum must be the total length \( L_C \) of the curve. The result is
\[ \int_C ds = L_C. \quad (4.7) \]

In words, this states that the line integral along a curve \( C \) with an integrand of one is equal to the length of the curve \( C \). Consequently, the final result for our integral of the constant \( K \) is
\[ I = \int_C K \, ds = KL_C. \quad (4.8) \]

This is a very simple result, valid for any curve \( C \). Of course, if the length \( L_C \) of the curve is not known, it is still necessary to evaluate the line integral \( \int_C ds \), and this can still be a formidable task. If the length is already known, however, then no more effort is necessary. For example, if the curve \( C \) is a circle of radius \( R \), then we know that \( L_C \) is the circumference of the circle, which is \( 2\pi R \). Hence, the integral \( I \) in this case is simply \( I = K2\pi R \).

### 4.4 Evaluation By Definite Integrals

Similar to the situation with definite integrals, the defining formula for line integrals is usually not of much value for the evaluation of the integral. We need a more practical way. For simplicity, we restrict our attention to integrals over curves that lie in a plane, which we call the \( x-y \) plane. The first thing we need to do is learn how to evaluate the integrand on the curve \( C \). Then, we will learn how to convert the line integral into a definite integral along either the \( x \) or the \( y \) axis.

#### 4.4.1 Integrand Evaluated On \( C \)

First, we need to learn how to express the curve mathematically. There are several different methods for expressing a given curve. We will stick to one method in this book, which is to express one of the variables as a function of
the other on the curve. For example, we can specify $y$ as a function of $x$ along the curve as $x$ varies over a specified interval in the form

$$y = y(x)$$

for $a_x \leq x \leq b_x$, where $y(x)$ indicates a function of $x$, or we can specify $x$ as a function of $y$ along the curve as $y$ varies over a specified interval in the form

$$x = x(y)$$

for $a_y \leq y \leq b_y$, where $x(y)$ indicates a function of $y$. Then, the value of the function $f(x, y)$ on the curve $C$ for a given value of $x$ between $a_x$ and $b_x$ is $f[x, y(x)]$, which depends only on $x$. The symbol $f[x, y(x)]$ indicates that $y$ is replaced by the function $y(x)$ wherever $y$ occurs in the function $f(x, y)$. Likewise, the value of the function $f(x, y)$ on the curve $C$ for a given value of $y$ between $a_y$ and $b_y$ is $f[x(y), y]$, which depends only on $y$. Note that once we know the equation for the curve $C$, then we need to specify only one of the coordinates (either $x$ or $y$) to evaluate the function $f(x, y)$ on the curve.

**Example 4.1** To clarify what is meant by the symbol $f[x, y(x)]$ above, consider an example where $f(x, y) = x \sin (ky) + y^3$ is to be evaluated on the curve $C$ defined by

$$y = Ax^2$$

for $0 \leq x \leq 5$. Then, to evaluate $f[x, y(x)]$, we need to substitute $y(x) = Ax^2$ in for $y$ wherever it appears in the function $f(x, y)$. The result is

$$f[x, y(x)] = x \sin (kAx^2) + A^3x^6,$$

for $0 \leq x \leq 5$. Note that the result depends only on $x$. This is because we are evaluating $f(x, y)$ on the curve for a given value of $x$. The value of $y$ on the curve for that value of $x$ is determined by the equation for $y$ as a function of $x$ for the curve.

The same curve $C$ can be expressed by the equation

$$x = \sqrt[4]{y/A}$$

for $0 \leq y \leq 25A$, since this equation is just the inverse of Eqn. (4.9) obtained by solving the latter equation for $x$ in terms of $y$. To use this equation to express $f(x, y)$ on the curve $C$ as a function of $y$, we substitute it in for $x$ everywhere it occurs in the expression for $f(x, y)$ to obtain

$$f[x(y), y] = \sqrt[4]{y/A} \sin (ky) + y^3$$

Another way to express the curve mathematically is to write $x$ as a function of a parameter $t$ and $y$ as another function of the same parameter on the curve as $t$ varies over some specified interval. This method is more general than the one we are using because it can easily be extended to curves that do not lie in a plane. The less general method was chosen for this book because it is easier to understand and use for our purposes.
for \(0 \leq y \leq 25\). Now, we have the function \(f(x,y)\) evaluated on the curve \(C\) as a function of \(y\) only.

**Example 4.2** For an example of a curve \(C\) which is initially specified by geometry rather than by an equation, consider the quarter of a circle in the first quadrant of the \(x,y\) plane with radius \(R\) and center at the origin. Then, from our knowledge of analytical geometry, we know the the equation for \(C\) is \(y = \sqrt{R^2 - x^2}\) with \(a_x = 0\) and \(b_x = R\), or \(x = \sqrt{R^2 - y^2}\) with \(a_y = 0\) and \(b_y = R\). Suppose the function \(f\) is \(f(x,y) = xy^2\), and we want to evaluate it on the curve \(C\) expressed in terms of only one variable.

First, let’s express it in terms of the variable \(x\). Then, we need to obtain \(f[x, y(x)]\) with \(y(x) = \sqrt{R^2 - x^2}\). We have

\[
f[x, y(x)] = x\sqrt{R^2 - x^2} = x(R^2 - x^2)
\]

for \(0 \leq x \leq R\).

Second, let’s express it in terms of the variable \(y\). Substituting \(x(y) = \sqrt{R^2 - y^2}\) into \(f(x,y)\) for \(x\), we have

\[
f[x(y), y] = \sqrt{R^2 - y^2} y^2
\]

for \(0 \leq y \leq R\).

### 4.4.2 Conversion to a Definite Integral

The first step in the conversion of the line integral into a definite integral along a coordinate axis is to relate the element of integration \(ds\) of the line integral to the element of integration \(dx\) or \(dy\) of the definite integral. When the curve segments \(\Delta s_i\) are sufficiently small, each segment is approximately a straight line, as indicated in Figure 4.3. Referring to the figure, we see that when we move a tiny distance \(\Delta s_i\) along the curve, the \(x\) coordinate moves a distance \(\Delta x_i\) and the \(y\) coordinate moves a distance \(\Delta y_i\), which are related (by the Pythagorean Theorem) by

\[
\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}. \quad (4.11)
\]

The increments \(\Delta x_i\) and \(\Delta y_i\) are not independent of each other: they are each determined by the distance \(\Delta s_i\) and the slope of the curve at that point. In particular, if \(C\) is given by \(y = y(x)\) for \(a_x \leq x \leq b_x\), then the slope of the curve at any value of \(x\) in that interval is equal to the derivative of \(y(x)\) with respect to \(x\) at that value of \(x\). Hence, \(\Delta y_i\) is given as a function of \(x\) by

\[
\Delta y_i = \frac{dy(x)}{dx} \Delta x_i. \quad (4.12)
\]

for sufficiently small \(\Delta x_i\). Likewise, if \(C\) is given by \(x = x(y)\) for \(a_y \leq y \leq b_y\), then the slope of the curve at any value of \(y\) in that interval is given by the
derivative of \( x(y) \) with respect to \( y \) at that value of \( y \). Hence, \( \Delta x_i \) is given as a function of \( y \) by
\[
\Delta x_i = \frac{dx(y)}{dy} \Delta y_i.
\] (4.13)
for sufficiently small \( \Delta y_i \). Combination of these formulas with Eqn. (4.11) yields
\[
\Delta s_i = \sqrt{1 + \left[ \frac{dy(x)}{dx} \right]^2} \Delta x_i,
\] (4.14)
\[
\Delta s_i = \sqrt{1 + \left[ \frac{dx(y)}{dy} \right]^2} \Delta y_i,
\] (4.15)
for sufficiently small \( \Delta x_i \) and \( \Delta y_i \). Recall that \( \Delta s_i \) is always positive. Hence, we must always choose \( \Delta x_i \) and \( \Delta y_i \) to be positive when using these formulas.

Since \( \Delta x_i \) and \( \Delta y_i \) both tend to zero when \( \Delta s_i \) does, we can substitute the latter two equations into the defining formula for the line integral Eqn. (4.2) to obtain
\[
\int_C f(x, y) \, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f[x_i, y(x_i)] \sqrt{1 + \left[ \frac{dy(x)}{dx} \right]^2} \Delta x_i
\] (4.16)
for \( a_x \leq x \leq b_x \), and
\[
\int_C f(x, y) \, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f[x(y_i), y_i] \sqrt{1 + \left[ \frac{dx(y)}{dy} \right]^2} \Delta y_i
\] (4.17)
for \( a_y \leq y \leq b_y \). In both of these equations, we have evaluated the integrand \( f(x,y) \) of the line integral on the curve \( C \) as described in the previous subsection, Sec. 4.4.1. Comparison of these equations with the defining equation for the definite integral Eqn. (2.2), shows that their right-hand-sides are definite integrals so that the equations become

\[
\int_C f(x,y) \, ds = \int_{a_x}^{b_x} f[x,y(x)] \sqrt{1 + \left( \frac{dy(x)}{dx} \right)^2} \, dx \quad (4.18)
\]
for \( b_x > a_x \),

\[
\int_C f(x,y) \, ds = \int_{a_y}^{b_y} f[x(y),y] \sqrt{1 + \left( \frac{dx(y)}{dy} \right)^2} \, dy \quad (4.19)
\]
for \( b_y > a_y \).

We have now expressed our line integral in terms of two different ordinary definite integrals. This means that we can evaluate the line integral by simply evaluating the appropriate definite integral. Note that regardless of the direction that we traverse \( C \) when we are doing the line integral, we always integrate in the positive \( x \) direction when we use the right-hand side of Eqn. (4.18) and in the positive \( y \) direction when we use the right-hand side of Eqn (4.19). Note also that we can not use the function \( y = y(x) \) to describe \( C \) in regions where \( C \) is parallel to the \( y \) axis because the derivative of \( y(x) \) with respect to \( x \) is infinite there. In such regions, we need to use \( x = x(y) \) to describe \( C \). For similar reasons, we need to use \( y = y(x) \) to describe \( C \) in regions where \( C \) is parallel to the \( x \) axis. Hence, if the curve \( C \) has one or more regions where it is parallel to the \( y \) axis, and also has one or more regions where it is parallel to the \( x \) axis as well, then the curve must be broken up into two or more segments so that the \( y \) integration can be used over some of the segments and the \( x \) integration can be used over the other ones.\(^3\)

Recall that we have been assuming all along that the curve \( C \) is smooth. Hence, another situation which requires breaking up the curve into different sections is when the slope of the curve changes discontinuously at isolated points. An example is worked out in detail in Sample Problem 4.1.

**Sample Problem 4.1** Integrate the function

\[ f(x,y) = Ax^2y^3 \]
around the closed curve \( C \), which is the triangle shown in Figure. 4.4 The quantities \( A, x_0, \) and \( y_0 \) are constants.

**Solution:**
**Givens:** \( A, x_0, y_0. \)

---

\(^3\)Exceptions to these statements sometime occur when the regions are isolate points. An example when it is still possible to apply Eqn. (4.18), even when \( C \) includes a point where the derivative of \( y(x) \) is infinite, is discussed in detail in Sample Problem 4.2.
Since the closed curve $C$ isn’t smooth at the vertexes of the triangle, we need to break it up into three open sub-curves $C_1$, $C_2$, and $C_3$ as follows:

- $C_1 : y(x) = \frac{y_0}{x_0} x$ for $0 \leq x \leq x_0$,
- $C_2 : y(x) = y_0$ for $x_0 \geq x \geq 0$,
- $C_3 : x(y) = 0$ for $y_0 \geq y \geq 0$.

The directions of the sub-curves are indicated in the figure and in the order of each inequality that specifies the domain of the variable for the corresponding sub-curve. Recall, though, that the direction of the curve doesn’t effect the value of the line integral, so it is of no real consequence to our problem.

The line integral $I$ around $C$ can be expressed as the sum of the integrals over each sub-curve as

$$I = I_1 + I_2 + I_3.$$ 

Here, $I_1$ is the line integral over $C_1$ given by Eqn. (4.18) as

$$I_1 = \int_0^{x_0} f[x, y(x)] \sqrt{1 + \left[ \frac{dy(x)}{dx} \right]^2} \, dx,$$

where $y(x) = \frac{y_0}{x_0} x$ and $\frac{dy(x)}{dx} = \frac{y_0}{x_0}$. With these values, we find

$$f[x, y(x)] = Ax^2 y(x)^3 = Ax^2 \left( \frac{y_0}{x_0} x \right)^3 = A \left( \frac{y_0}{x_0} \right)^3 x^5$$

and

$$\sqrt{1 + \left[ \frac{dy(x)}{dx} \right]^2} = \sqrt{1 + \left[ \frac{y_0}{x_0} \right]^2}.$$
Hence,

\[ I_1 = A \left( \frac{y_0}{x_0} \right)^3 \sqrt{1 + \left( \frac{y_0}{x_0} \right)^2} \int_0^{x_0} x^5 \, dx = A \left( \frac{y_0}{x_0} \right)^3 \sqrt{1 + \left( \frac{y_0}{x_0} \right)^2} \frac{x_0^6}{6}. \]

\( I_2 \) is the integral over \( C_2 \) given by Eqn. (4.18) as

\[ I_2 = \int_0^{x_0} f[x, y(x)] \sqrt{1 + \left( \frac{dy(x)}{dx} \right)^2} \, dx, \]

where \( y = y_0 \) and \( \frac{dy(x)}{dx} = 0 \). With these values, we find

\[ f[x, y(x)] = Ax^2y_0^3 \]

and

\[ \sqrt{1 + \left( \frac{dy(x)}{dx} \right)^2} = 1, \]

giving

\[ I_2 = A \int_0^{x_0} x^2y_0^3 \, dx = A \frac{x_0^3y_0^3}{3}. \]

Finally, \( I_3 \) is the integral over \( C_3 \) given by Eqn. (4.19) as

\[ I_3 = \int_0^{y_0} f[x(y), y] \sqrt{1 + \left( \frac{dx(y)}{dy} \right)^2} \, dy, \]

where \( f[x(y), y] = A0^2y^3 = 0 \) all along \( C_3 \). Hence, \( I_3 = 0 \).

The final result for the desired line integral over the whole triangle is found by adding the results for the individual sides to obtain

\[ I = \frac{x_0^3y_0^3}{3} \left[ \frac{1}{2} \sqrt{1 + \left( \frac{y_0}{x_0} \right)^2} + 1 \right]. \]

**Sample Problem 4.2** Let \( C \) be the upper-right quadrant of a circle of radius \( R \) centered at the origin as shown in Figure 4.5 with the direction as shown. Integrate the function \( f(x, y) = x^2 + y^2 \) along \( C \).

**Solution:**

**Givens:** \( R \).

\footnote{Note here that the integration goes from 0 to \( x_0 \) even though the direction of \( C_2 \) is in the opposite direction. This is because the \( x \) (or \( y \)) integral representing the line integral must go in the positive direction independent of the direction of the curve. See Sec. 4.4.2.}
Figure 4.5: The open curve $C$ used for Sample Problem 4.2.

Note from the figure that curve is parallel to the $x$ axis at its highest point and is parallel to the $y$ axis at its lowest point. Hence, it appears that neither Eqn. (4.18) nor Eqn. (4.19) can be applied over the entire curve. One way to deal with this difficulty would be to break the curve into two parts and use Eqn. (4.18) for the upper part and Eqn. (4.19) for the lower part. The addition of the two results would give the result for the integral over the entire curve. That is a good approach to follow in general, but it is shown below that it is not necessary to do that in this special case. Either of Eqn. (4.18) or Eqn. (4.19) can be used over the entire curve successfully. Eqn. (4.18) is applied here. For that purpose, we express the curve $C$ in the form $y = y(x)$ where $y(x) = \sqrt{R^2 - x^2}$ for $0 \leq x \leq R$.

Since
\[
\frac{dy(x)}{dx} = -\frac{x}{\sqrt{R^2 - x^2}},
\]

it follows that
\[
\sqrt{1 + \left(\frac{dy(x)}{dx}\right)^2} = \frac{R}{\sqrt{R^2 - x^2}},
\]

Moreover, we have
\[
f[x, y(x)] = x^2 + [y(x)]^2 = x^2 + \left(\sqrt{R^2 - x^2}\right)^2 = R^2.
\]

Hence, Eqn. (4.18) for our contour integral over the circular arc in Figure 4.5 becomes
\[
I = \int_C f(x, y) \, ds = \int_0^R f[x, y(x)] \sqrt{1 + \left(\frac{dy(x)}{dx}\right)^2} \, dx = \int_0^R \frac{R^3}{\sqrt{R^2 - x^2}} \, dx.
\]
CHAPTER 4. LINE INTEGRALS

Notice that the integrand in this equation is infinite at the upper limit of integration. We say that the integrand has a singularity at \( x = R \). This is the consequence of the curve being parallel to the \( y \) axis at that point. This singularity is not really a problem, however, because it tends toward infinity so slowly as \( x \) approaches \( R \) that it does not cause the integral to tend toward infinity. If we integrate from 0 to \( r \) where \( 0 \leq r \leq R \), and then take the limit as \( r \) tends to \( R \), we get a finite result. This gives meaning to this "improper integral".

It is not necessary to follow that procedure in this case, however, because we can "remove" the singularity by making a change of the variable of integration. For that purpose, we introduce a new variable of integration \( \theta \) by setting \( x = R \cos \theta \), giving \( dx = \frac{dx}{d\theta} d\theta = -R \sin \theta d\theta \). The integral then reduces to

\[
I = \int_{\theta_1}^{\theta_2} R^3 \frac{d\theta}{\sqrt{R^2 - R^2 \cos^2 \theta}} \left( -R \sin \theta d\theta \right) = \int_{\theta_1}^{\theta_2} R^3 \, d\theta.
\]

Since there is no longer a singularity in the integrand, we see that the integral has a finite value. (When a singularity can be removed in this way, we say that it is a removable singularity.) The final result for our integral is

\[
I = R^3 \frac{\pi}{2}.
\]

4.5 Line Integrals Along Circular Arcs

A shorthand way that we often use to calculate line integrals on circular arcs is called "conversion to polar coordinates". Suppose the arc has radius \( R \) and runs from \( \theta = \theta_1 \) to \( \theta = \theta_2 \) where \( \theta_1 < \theta_2 \). In this case, it is convenient to express \( x \) and \( y \) in polar coordinates \( R \) and \( \theta \) as follows:

\[
x = R \cos \theta,
\]

\[
y = R \sin \theta.
\]

The geometry of this transformation is indicated in Figure 4.6, where it is shown that \( \theta \) is the angle which the line from the origin to the point \((x, y)\) makes with the \( x \) axis. As can be seen from the figure, the distance \( x \) can be thought of being the projection of the former line onto the \( x \) axis, whereas the distance \( y \) can be thought of as being the projection of the same line onto the \( y \) axis. In order to express \( ds \) along the circular arc in terms of \( R \) and \( \theta \), we make use of the geometry shown in Figure 4.7. We see that, provided that the angle \( \theta \) is measured in radians, the element of arc length \( ds \) along the circular arc is related to the change in angle \( d\theta \) by the relation

\[
ds = R \, d\theta.
\]

To keep \( ds \) positive, we must always integrate in the positive \( \theta \) direction to keep \( d\theta \) positive. The line integral along the curve \( C \) now becomes

\[
I = \int_C f(x, y) \, ds = R \int_{\theta_1}^{\theta_2} f(x(\theta), y(\theta)) \, d\theta.
\]
4.5. **LINE INTEGRALS ALONG CIRCULAR ARCS**

4.5.1.

**Diagram 4.6:** Geometry of polar coordinates

**Diagram 4.7:** Geometry to relate $\Delta s$ to $\Delta \theta$ along a circular arc.

with $\theta_1 < \theta_2$. Usually, the function $f(x, y)$ to be integrated is specified as a function $F(\theta)$ of $\theta$, so the integral of interest can be written

$$I = \int_C F(\theta) \, d\theta.$$ 

We have again expressed the line integral as a definite integral, which allows us to evaluate the line integral by doing regular integration. Keep in mind, however, that this method is applicable only when the curve $C$ follows a circular arc. If the curve is circular, it is often much more convenient to use polar coordinates and apply the above equations. This is illustrated in Sample Problem 4.3.

**Sample Problem 4.3** Let $C$ be the upper-right quadrant of a circle of radius $R$ centered at the origin as shown in Figure 4.5 with the direction as shown. Integrate the function $f(x, y) = x^2 + y^2$ along $C$.

**Solution:**

Givens: $R$.

This is the same problem as the one we solved in Sample Problem 4.2. This time, we solve it using polar coordinates. In order for $x = R \cos \theta$ to vary over the interval from 0 to $R$, the new variable of integration $\theta$ must vary from $\frac{\pi}{2}$ to 0. However, we must integrate in the positive $\theta$ direction to keep $d\theta$ positive. Hence, our line integral becomes

$$I = \int_C f(x, y) \, ds = R \int_0^{\frac{\pi}{2}} f[x(\theta), y(\theta)] \, d\theta$$
Converting the integrand to polar coordinates, we have
\[ f([x(\theta), y(\theta)]) = x(\theta)^2 + y(\theta)^2 = (R \cos \theta)^2 + (R \sin \theta)^2 = R^2(\cos^2 \theta + \sin^2 \theta) = R^2. \]
Hence, \( I \) becomes
\[ I = R \int_0^\frac{\pi}{2} R^2 \, d\theta = R^3 \frac{\pi}{2}. \]
This is the same result as we obtained in Sample Problem 4.2 with much more effort.

4.6 Forces Due to Line Charges

One of the reasons we have studied line integrals is that we need to use them whenever we need to calculate the force \( \vec{F}_q \) exerted on a point charge \( q \) by an extended charge that is distributed over a curve \( C \). We are now in a position to be able to perform such calculations provided that \( C \) lies within a plane (which we take to be the x-y plane).

We learned how to approach the problem in Sec. 3.6.4. The force is calculated by a charge integral in the form
\[ \vec{F}_q = kq \int_{dQ} \frac{\vec{r}}{r^3} dQ, \]
where
\[ \vec{r} = (x_p - x)\hat{i} + (y_p - y)\hat{j} + (z_p - z)\hat{k} \]
is the vector from the element of integration \( dQ \) located at \((x, y, z)\) to the point charge \( q \) located at the point \((x_p, y_p, z_p)\). Its magnitude is
\[ r = \sqrt{(x_p - x)^2 + (y_p - y)^2 + (z_p - z)^2}. \]

If the charge is distributed over a curve \( C \) with linear charge density (charge per unit length) \( \lambda(s) \), then the element \( dQ \) of charge integration is related to element \( ds \) of line integration by\(^5\)
\[ dQ = \lambda(s) \, ds \]
(see Sec. 3.5). Consequently, the charge integral can be converted into a line integral, giving
\[ \vec{F}_q = kq \int_C \frac{\vec{r}}{r^3} \lambda(s) \, ds. \]

Since the above line integral involves the adding of many vectors, it is necessary to break the vector equation down into its three scalar components in order to be able to perform the integration. The result is
\[ (F_q)_x = kq \int_C \frac{x_p - x}{r^3} \lambda(s) \, ds, \]
\(^5\)Recall that \( ds \) is always positive, which guarantees the \( dQ \) has the same sign as \( \lambda \).
4.6. FORCES DUE TO LINE CHARGES

\[ (F_q)_y = kq \int_C \frac{y_P - y}{r^3} \lambda(s) \, ds, \]
\[ (F_q)_z = kq \int_C \frac{z_P - z}{r^3} \lambda(s) \, ds. \]

These equations present complete expressions for the force on the point charge \( q \). If we know the charge \( q \), its location, the curve \( C \), and the linear charge density \( \lambda(s) \), then the evaluation of the three line integrals is all that remains to be done.

**Sample Problem 4.4** Consider a very thin circular ring of radius \( R \), lying in the \( x \)-\( y \) plane with its center at the origin. It has linear charge density \( \lambda(s) = \lambda_0 \sin \theta \), where \( \lambda_0 \) is a constant, and \( \theta \) is the angle that a line from the origin to the element of integration \( ds \) on the ring makes with the \( x \) axis (see Figure 4.8). Determine the force \( \vec{F}_q \) that the ring exerts on a point charge \( q \) located on the \( z \) axis at \( z = z_p \).

**Solution:**

**Givens:** \( R, \lambda_0, z = 0, q, x_p = 0, y_p = 0, z_p \).

Since the curve \( C \) is circular, we use polar coordinates and set \( x = R \cos \theta \), \( y = R \sin \theta \), and \( ds = R \, d\theta \). Substitution of the polar expressions for \( x \) and \( y \), and the zero values for \( z, x_p \), and \( y_p \) into the above equation for \( r \) gives

\[ r = \sqrt{R^2 + z_p^2}. \]

Starting with the expression above for the \( x \) component of the force, we find
with the above substitutions
\[(F_q)_x = kq \int_C \frac{-y}{r^3} \lambda_0 \sin \theta \, ds = -kq \frac{\lambda_0 R^2}{(R^2 + z_p^2)^{\frac{3}{2}}} \int_0^{2\pi} \cos \theta \sin \theta \, d\theta.\]

The final integral is easily seen to be zero by studying the cyclic behavior of the integrand over the range of integration, or by using the u-substitution \(u = \sin \theta\) to evaluate the integral (see Sec. 2.4.1).

Making the same substitutions in the expression for the \(y\) component of the force, we obtain
\[(F_q)_y = kq \int_C \frac{-y}{r^3} \lambda_0 \sin \theta \, ds = -kq \frac{\lambda_0 R^2}{(R^2 + z_p^2)^{\frac{3}{2}}} \int_0^{2\pi} \sin^2 \theta \, d\theta.\]

To evaluate the final integral, we make use of the half-angle formula
\[\sin \frac{\phi}{2} = \sqrt{\frac{1}{2}(1 - \cos \phi)}\]
to see that
\[\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta).\]

With this expression substituted into the above final integral, we have two definite integrals to evaluate. The first is the integral of \(\frac{1}{2}\), which yields \(\pi\), and the other is the integral of \(-\frac{1}{2} \cos 2\theta\), which is zero. Hence, the final expression for the \(y\) component of the force is
\[(F_q)_y = -kq \frac{\lambda_0 R^2}{(R^2 + z_p^2)^{\frac{3}{2}}} \pi.\]

Finally, we make the same substitutions in the expression for the \(z\) component of the force to obtain
\[(F_q)_z = kq \int_C \frac{z_p}{r^3} \lambda_0 \sin \theta \, ds = kq \frac{\lambda_0 R z_p}{(R^2 + z_p^2)^{\frac{3}{2}}} \int_0^{2\pi} \sin \theta \, d\theta = 0.\]

Since the only component of the force which is not zero is the \(y\) component, the force vector can be written as
\[\vec{F}_q = -kq \frac{\lambda_0 R^2}{(R^2 + z_p^2)^{\frac{3}{2}}} \pi \hat{j}.\]
4.6. FORCES DUE TO LINE CHARGES

PROBLEMS

1. Evaluate the line integral

\[ I = \int_{C} x^3 y \sqrt{x^2 + y^2} \, ds, \]

where \( C \) is the curve shown in Figure 4.9. Only one solution is required. It is a symbolic solution in that it involves the symbol \( d \), which is shown in Figure 4.9 as a given positive constant. Do not assign symbols to the ends of the curve \( C \). Keep them at the numerical values indicated in the figure. (Solution check: the numerical value of \( I \) for \( d = 3 \) m is \( I = 9.82 \times 10^3 \) m\(^6\).)

![Figure 4.9: Problem 1](image_url)

2. Evaluate the line integral

\[ I = \int_{C} x^2 y \, ds, \]

where \( C \) is the curve shown in Figure 4.10. Only one solution is required. It is a symbolic solution in terms of the two given positive constants \( p \) and \( q \). (Solution check: \( I = -3.61 \) m\(^4\) for \( p = 2 \) m and \( q = 3 \) m.)

![Figure 4.10: Problem 2](image_url)

3. An electric wire suspended between two towers forms a catenary modeled by the equation

\[ y = x_0 \cosh \frac{x}{x_0} \]

for \(-x_0 \leq x \leq x_0\), where \( x_0 \) is a given positive constant. The term “cosh” indicates the hyperbolic cosine. Find a symbolic solution for the length
4. Evaluate the line integral
\[ I = \int_C \left( x^2 + y^2 \right)^{3/2} y^2 \, ds, \]
where \( C \) is the semicircular arc shown in Figure 4.12. Give one solution in terms of the positive constant \( R \). (Solution check: \( I = 101 \, m^6 \) for \( R = 2 \, m \).)

5. Consider a uniformly charged, very thin circular ring of charge \( Q \) and radius \( R \), lying in the \( x-y \) plane with its center at the origin. Determine the force \( \vec{F}_q \) that the ring exerts on a point charge \( q \) located on the \( z \) axis at \( z = z_p \). (Solution check: \( \vec{F}_q = 1.73 \times 10^9 \hat{k} \, N \) for \( Q = 2 \, C, q = 3 \, C, R = 3 \, m, \) and \( z_p = 4 \, m \).)

6. (Supplemental problem for students who have studied the chapter on electric potential in their physics text.) Consider a very-thin plastic thread that lies in the \( x-y \) plane, following the curve \( y = Ax^2 \) for \( 0 \leq x \leq x_0 \), where \( A \) and \( x_0 \) are positive constants. The thread is charged with linear charge density (charge per unit length) which varies with position according to \( \lambda(x) = Bx^2 \sqrt{1 + A^2 x^2} \), where \( B \) is a positive constant. There are no other charges present. With the potential \( V \) equal to zero at infinity, obtain an expression for \( V \) at the origin. (Solution check: The numerical value with \( A = 2.00 \, m^{-1}, B = 3.00 \, C \, m^{-3} \), and \( x_0 = 4.00 \, m \) is \( 2.31 \times 10^{12} \) volts.)
Chapter 5

Double Integrals

5.1 Introduction

This chapter is concerned with the integration of a function over a flat surface area, which is taken to lie in the $x$-$y$ plane. The integrals of interest are often referred to as area integrals, which can be written in the form

$$I = \int_A f(x, y) \, dA,$$  \hspace{1cm} (5.1)

where the $A$ at the bottom of the integral sign indicates the two-dimensional region of the $x$-$y$ plane over which we are integrating. This notation can lead to confusion because it appears very much like the notation for a line integral and does not show explicitly that the integration involves two dimensions rather than just one. A notation that acknowledges the two-dimensionality of the integration expresses the area integral as

$$I = \iint_A f(x, y) \, dA.$$  \hspace{1cm} (5.2)

When written this way, the area integral is referred to as a double integral.

These integrals make up a special case of the more general surface integrals treated in the following chapter, which covers integration over surfaces which can be curved. The special case treated here deserves its own chapter for two reasons. First, flat surfaces appear frequently in applications, and their treatment is much simpler than that of curved surfaces. Secondly, the method discussed in the following chapter for evaluating surface integrals over curved surfaces make use of the area integrals discussed here. Since all four of Maxwell’s equations (the key equations in the theory of electromagnetism) involve surface integrals of either flat or curved surfaces, understanding the material of this chapter is essential for success in the study of electricity and magnetism.
5.2 Definition

The area integral is defined by the same procedure as the one used to define the definite integral in Section 2.2 with one change. Instead of considering the interval from \(a\) to \(b\) along the \(x\) axis and partitioning it into tiny sub-intervals \(\Delta x_i\), we consider the two-dimensional region \(A\) in the \(x-y\) plane and partition it into tiny two-dimensional sections of area \(\Delta A_i\). The complete procedure is as follows.

1. Break the region \(A\) into \(n\) tiny sections. The area of the \(i^{th}\) section is \(\Delta A_i\) with \(i\) ranging from 1 to \(n\), and all of these areas are taken to be positive. We do the partition so that the areas of all of the tiny sections exactly cover the region \(A\), and they all tend towards zero simultaneously as the number \(n\) of the sections increases towards infinity.\(^1\)

2. Evaluate the integrand at the location \((x_i, y_i)\) of each section and multiply it times the area of the section \(\Delta A_i\).

3. Add together all \(n\) of the above products of the multiplications.

4. Take the limit of the sum as the number \(n\) of tiny sections tends to infinity.

The result of this process is the area integral expressed in Eqn. (5.1). After doing the first step described above, all of the remaining steps can be expressed mathematically by a single equation

\[
\int_A f(x, y) \, dA = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i. \tag{5.3}
\]

Equation (5.3), taken together with the subdivision of \(A\) into the \(n\) tiny sections of area \(\Delta A_i\), gives the mathematical definition of the area integral.

Because the definitions of area integrals and definite integrals are so similar, they share many common properties. For example, if the integrand \(f(x, y)\) includes a constant factor \(K\) so that \(f(x, y) = Kg(x, y, z)\), then we have

\[
\int_A Kg(x, y) \, dA = K \lim_{n \to \infty} \sum_{i=1}^{n} g(x_i, y_i) \Delta A_i = K \int_A g(x, y) \, dA, \tag{5.4}
\]

where the constant \(K\) has been factored outside of the sum because it is included in every term. The result is that the constant factor in the integrand can be moved outside of the integral sign just as it can be done with definite integrals.

Two more basic properties of area integrals that are shared with definite integrals are:

\[
\int_A [f(x, y) + g(x, y)] \, dA = \int_A f(x, y) \, dA + \int_A g(x, y) \, dA \tag{5.5}
\]

\(^1\)Since two-dimensional regions can take very complicated shapes, it may be necessary to limit the choice of shapes permitted for \(\Delta A_i\) in order to avoid problems with the sums and limits involved in the definition of the area integral. In later sections of this chapter, we choose specific shapes to use which are known to cause no problems.
and
\[ \int_{A_1 + A_2} f(x, y) \, dA = \int_{A_1} f(x, y) \, dA + \int_{A_2} f(x, y) \, dA. \tag{5.6} \]
These properties again follow directly from the definition of area integrals given in Eqn. (5.3).

5.3 Area Integral of a Constant

A very important special case of area integrals to keep in mind is the one with integrand \( f(x, y, z) = K \) where \( K \) is a constant. Factoring the constant outside of the integral sign, we have
\[
I = \int_A K \, dA = K \int_A dA.
\]
According to the defining equation for area integrals, Eqn. (5.3), the area integral on the right-hand-side of the above equation is
\[
\int_A dA = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta A_i.
\]
Since the sum on the right-hand-side of the above equation adds together the areas of all of the tiny sections that make up \( A \), the result of the sum must be the total area \( A \) of the region \( A \). The result is
\[
\int_A dA = A. \tag{5.7}
\]
In words, this states that the area integral of a two-dimensional region \( A \) with an integrand of one is equal to the area \( A \) of the region \( A \). Consequently, the final result for our integral of the constant \( K \) is
\[
I = \int_A K \, dA = KA. \tag{5.8}
\]
This is a very simple result, valid for any flat, two-dimensional region \( A \). Of course, if the area \( A \) of the region \( A \) is not known, it is still necessary to evaluate the area integral \( \int_A dA \), and this can still be a formidable task. If the area is already known, however, then no more effort is necessary. For example, if the region \( A \) is a flat disc of radius \( R \), then we know that \( A \) is the area of a circle, which is \( \pi R^2 \). Hence, the integral \( I \) in this case is simply \( I = K \pi R^2 \).

5.4 Evaluation Using Cartesian Coordinates

As usual, it is not convenient to evaluate area integrals by using the defining equation Eqn. (5.3). Instead, we need to learn how to express the area integral
in terms of definite integrals. One way to do this is to choose the little areas to be rectangles with sides parallel to the $x$ and $y$ axes with lengths respectively $\Delta x_i$ and $\Delta y_i$, both of which are taken to be positive. Then, we have

$$\Delta A_i = \Delta x_i \Delta y_i.$$  \hfill (5.9)

Since some regions $A$ which have complicated shapes can prove to be very difficult to treat, we restrict our attention to two different types of shapes. The first is called “$y$-simple” and the other is called “$x$-simple”. The difference between these two different types is illustrated in Figures 5.1 and 5.2.

![Figure 5.1: Shape which is y-simple but is not x-simple.](image)

![Figure 5.2: Shape which is x-simple but is not y-simple.](image)

The feature that makes the shape in Figure 5.1 $y$-simple is that every line parallel to the $y$ axis that passes in and out of $A$, does so only once. This means that the points $(x, y)$ that are in $A$ are specified by the two inequalities

$$a_x \leq x \leq b_x$$  \hfill (5.10)

and

$$y_-(x) \leq y \leq y_+(x),$$  \hfill (5.11)

where $a_x$ and $b_x$ are constants that express the extreme values of $x$ within $A$, $y = y_+(x)$ is the equation for the upper boundary of $A$, and $y = y_-(x)$ is the equation for the lower boundary of $A$. Since $y_-(x) \leq y_+(x)$ for each value of $x$ within $A$, the upper and lower boundaries can touch each other but can never cross.

Likewise, the feature that makes the shape in Figure 5.2 $x$-simple is that every line parallel to the $x$ axis that passes in and out of $A$, does so only once. This means that the points $(x, y)$ in $A$ are specified by the two inequalities

$$a_y \leq y \leq b_y$$  \hfill (5.12)

and

$$x_-(y) \leq x \leq x_+(y),$$  \hfill (5.13)
5.4. EVALUATION USING CARTESIAN COORDINATES

where \( a_y \) and \( b_y \) are constants that specify the extreme values of \( y \) within \( A \), \( x = x_+(y) \) is the equation for the right boundary of \( A \), and \( x = x_-(y) \) is the equation for the left boundary of \( A \). Since \( x_-(y) \leq x_+(y) \) for each value of \( y \) within \( A \), the two boundaries can touch each other but can never cross.

All of the simple basic shapes we work with most frequently, such as rectangles, parallelograms, triangles, circles, and ellipses, are both \( x \)-simple and \( y \)-simple. Such shapes are said to be “regular”.

**Example 5.1** A simple example of a regular shape is the upper half of the circular disk of radius \( R \) centered at the origin as shown in Figure 5.3. The upper boundary is \( y = y_+(x) = \sqrt{R^2 - x^2} \) and the lower boundary is \( y = y_-(x) = 0 \) for \(-R \leq x \leq R \). Similarly, the left boundary is \( x = x_-(y) = -\sqrt{R^2 - y^2} \) and the right boundary is \( x = x_+(y) = \sqrt{R^2 - y^2} \) for \( 0 \leq y \leq R \).

\[ y = a = y_f(x) \]

\[ y = b = y_i(x) \]

5.4.1 \( y \)-simple Shapes

We deal first with area integrals over a region \( A \) that is \( y \)-simple. We convert the area integral into a pair of definite integrals by performing the operations indicated in the defining equation Eqn. (5.3) in a specific order. Consider the \( y \)-simple region \( A \) shown in Figure 5.4 with the upper boundary given by the equation \( y = y_+(x) \) and the lower boundary given by \( y = y_-(x) \). We begin adding the terms in the sum indicated in Eqn. (5.3) by choosing an arbitrary value of \( x_i \) and adding together the contributions of all of the elements \( A_i \) that have that value of \( x_i \). This will be the amount that the rectangle shaded in the figure contributes to the sum. The rectangle has a width of \( \Delta x_i \) and a height
equal to $y_+(x_i) - y_-(x_i)$. The value $Sum(x_i, n_j)$ of this contribution is

$$Sum(x_i, n_j) = \Delta x_i \sum_{j=1}^{n_j} f(x_i, y_j) \Delta y_j.$$  \hspace{1cm} (5.14)

The upper limit $n_j$ of the above sum is the number of $\Delta y_j$ elements that fit into the interval $y_+(x_i) - y_-(x_i)$ between the two curves at $x = x_i$. It is important to recognize that the sum in this equation is calculated with $x_i$ remaining constant and $y_j$ varying over the entire strip from $y_-(x_i)$ to $y_+(x_i)$.

We now take the limit as $n_j$ tends to infinity. From the defining equation for the definite integral Eqn. (2.2), we see that the result is $\Delta x_i$ times the definite integral of $f(x_i, y)$ with respect to $y$ from $y = y_-(x_i)$ to $y = y_+(x_i)$ with $x_i$ held constant, or

$$\lim_{n_j \to \infty} Sum(x_i, n_j) = \Delta x_i \int_{y_-(x_i)}^{y_+(x_i)} f(x_i, y) \, dy.$$ \hspace{1cm} (5.15)

It is important to keep in mind when using Eqn. (5.15) that $x_i$ must be held constant when performing the integration and evaluating the result at the limits of integration.

The definite integral on the right-hand-side this equation is not a function of $y$ because the $y$ coordinate has been integrated out, but it is a function of $x_i$ for 2 reasons. One is that the integrand is a function of $x_i$ and the other is that the limits of integration are both functions of $x_i$.

We can now complete the sum in the defining equation Eqn. (5.3) for the area integral by adding all of the terms specified in Eqn. (5.15) for $i$ ranging
from 1 to \( n_i \) and then taking the limit as \( n_i \) tends to infinity. The result is

\[
\int_A f(x, y) \, dA = \lim_{n_i \to \infty} \sum_{i=1}^{n_i} \Delta x_i \int_{y_i(x_i)}^{y+(x_i)} f(x_i, y) \, dy. \tag{5.16}
\]

Comparison of the sum in Eqn. (5.16) with the definition of the definite integral given in Eqn. (2.2) shows that the sum becomes a definite integral in the limit as \( n_i \) tends to infinity. The integrand of that integral is an integral itself

\[
\int_{y-(x)}^{y+(x)} f(x, y) \, dy.
\]

Consequently, Eqn. (5.16) becomes

\[
\int_A f(x, y) \, dA = \int_{a_x}^{b_x} \left[ \int_{y-(x)}^{y+(x)} f(x, y) \, dy \right] \, dx. \tag{5.17}
\]

We have now expressed our double integral in terms of two definite integrals, one done after the other. As the notation in the equation indicates, we first do the definite integral with respect to \( y \) in the square brackets considering \( x \) to be a constant. Upon completion of that integral, we then evaluate the resulting definite integral with respect to \( x \). Such repeated integrals are called iterated integrals. They are usually written without the brackets that appear in the above equation, in which case, the double integral notation is usually used for the area integral on the left-hand side. We then have

\[
\int \int_A f(x, y) \, dA = \int_{a_x}^{b_x} \int_{y-(x)}^{y+(x)} f(x, y) \, dy \, dx. \tag{5.18}
\]

With this notation, the inner integral (the \( y \) integral in this case) is always done first and the outer integral is done last.\(^2\) It is important to remember when using these iterated integrals to calculate an area integral that both the \( y \) and the \( x \) integrals must always be done in the positive directions.

**Example 5.2** Since the shape of the area \( A \) in Example 5.1 is \( y \)-simple, we can write the area integral over \( A \) of the integrand \( f(x, y) \) as

\[
I = \int_A f(x, y) \, dA = \int_{-R}^{R} \int_{0}^{\sqrt{R^2-x^2}} f(x, y) \, dy \, dx.
\]

**Sample Problem 5.1** Find the area integral of the function \( f(x, y) = Bx^2y^3 \) (where \( B \) is a constant) over the area of the triangle given in Figure 5.5.

**Solution:**

**Givens:** \( B, x_0, y_0 \).
Note that this problem involves the same function and geometry as Sample Problem 4.1, but it is a very different problem because it requires doing an area integral over area of the triangle instead of a line integral around the perimeter of the triangle.

Since the region $A$ consists of all of the points satisfying

$$0 \leq x \leq x_0$$

and

$$\frac{y_0}{x_0} x \leq y \leq y_0,$$

we see that it is y-simple. Application of Eqn. (5.18) gives

$$I = \int \int_A Bx^2 y^3 \, dA = \int_0^{x_0} \int_{y_0 x/x_0}^{y_0} Bx^2 y^3 \, dy \, dx$$

Evaluating the $y$ integral while keeping $x$ constant, we obtain

$$I = B \int_0^{x_0} x^2 \left( y_0^4 - \left( \frac{y_0 x}{x_0} \right)^4 \right) \, dx.$$  

Finally, we evaluate the $x$ integral to obtain

$$I = B \frac{y_0^4}{4} \left( \frac{x_0^3}{3} - \frac{x_0^7}{7x_0^4} \right) = B \frac{x_0^3 y_0^4}{21}.$$  

\[2\text{Another way to say this is "The integral that is to be done first is the one whose differential appears first (i.e. $dy \, dx$ indicates that the $y$ integral is to be done first)."} \]
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5.4.2 x-simple Shapes

Now, consider the x-simple region $A$ shown in Figure 5.6. The results that correspond to those obtained in the previous section for y-simple shapes can be obtained using the same approach with minor modifications. Instead, we follow the short-hand approach which is usually used in practice. We first choose an arbitrary line of constant $y$ which passes through $A$ and then integrate $f(x, y)$ along that line from the left boundary $x_{-}(y)$ to the right boundary $x_{+}(y)$. After multiplying the result by $dy$, we have the contribution to the area integral due to the shaded rectangle in the figure

$$\left[ \int_{x_{-}(y)}^{x_{+}(y)} f(x, y) \, dx \right] dy.$$

We then integrate this result with respect to $y$ from its lower extreme position $a_y$ in $A$ to its upper extreme position $b_y$ to obtain the contribution to the area integral due to the entire area $A$. The result is the desired area integral

$$\int \int_{A} f(x, y) \, dA = \int_{a_y}^{b_y} \int_{x_{-}(y)}^{x_{+}(y)} f(x, y) \, dx \, dy. \quad (5.19)$$

Again, we have an iterated integral, but this time we do the $x$ integral first, followed by the $y$ integral.

**Example 5.3** Since the shape of the area $A$ in Example 5.1 is x-simple, we can write the area integral over $A$ of the integrand $f(x, y)$ as

$$I = \int_{A} f(x, y) \, dA = \int_{0}^{R} \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} f(x, y) \, dx \, dy.$$
Sample Problem 5.2  Find the area integral of the function $f(x, y) = x^2 + y^2$ over the area of the quarter circle given in Figure 5.7.

Solution:
Givens: $R$.

Note that this problem involves the same function and geometry as Sample Problem 4.2, but it is a very different problem because it requires doing an area integral over area of the quarter circle instead of a line integral around the perimeter of the quarter circle.

Since the points in $A$ satisfy

$$0 \leq y \leq R$$

and

$$0 \leq x \leq \sqrt{R^2 - y^2},$$

the area is x-simple. Hence, application of Eqn. (5.19) yields

$$I = \int \int_A f(x, y) \, dA = \int_0^R \int_0^{\sqrt{R^2 - y^2}} (x^2 + y^2) \, dx \, dy.$$ 

Evaluating the $x$ integral while holding $y$ constant gives

$$I = \int_0^R \left[ \frac{(R^2 - y^2)^{3/2}}{3} + y^2 \sqrt{R^2 - y^2} \right] dy.$$
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The integral in this equation is difficult to do\(^3\), but with the aid of an integral table[GR65, p. 86, integrals 2.272-2 and 2.271-2], we find

\[
I_a = \int_0^R y^2 \sqrt{R^2 - y^2} \, dy = \frac{R^4 \pi}{16},
\]

\[
I_b = \int_0^R \frac{(R^2 - y^2)^{3/2}}{3} \, dy = \frac{R^4 \pi}{16}.
\]

Hence, the final result for our area integral is

\[
I = I_a + I_b = \frac{R^4 \pi}{8}.
\]

5.4.3 Regular Shapes

Recall that regular shapes are those that are both x-simple and y-simple. Consider a double integral \(I\) over the two dimensional region \(A\) which has a regular shape

\[
I = \int \int_A f(x, y) \, dA. \quad (5.20)
\]

Since its shape is y-simple, the points \((x, y)\) within \(A\) are specified by the inequalities

\[
a_x \leq x \leq b_x
\]

and

\[
y_-(x) \leq y \leq y_+(x),
\]

where \(a_x\) and \(b_x\) are constants that express the extreme values of \(x\) within \(A\), \(y = y_+(x)\) is the equation for the upper boundary of \(A\), and \(y = y_-(x)\) is the equation for the lower boundary of \(A\). Hence, our area integral \(I\) can be written as the iterated integral

\[
I = \int_{a_x}^{b_x} \int_{y_-(x)}^{y_+(x)} f(x, y) \, dy \, dx. \quad (5.23)
\]

Similarly, since \(A\)'s shape is x-simple, the points \((x, y)\) within \(A\) are specified by the inequalities

\[
a_y \leq y \leq b_y
\]

and

\[
x_-(y) \leq x \leq x_+(y),
\]

where \(a_y\) and \(b_y\) are constants that express the extreme values of \(y\) within \(A\), \(x = x_+(y)\) is the equation for the right boundary of \(A\), and \(x = x_-(y)\) is the

\(^3\)It can be done by making the trigonometric substitution \(y = R \sin \theta\) and applying various trigonometric identities to manipulate the integrand to obtain an integrable form. The calculation is too lengthy to present here.
equation for the left boundary of $A$. Hence, our area integral $I$ can be written as the iterated integral

$$I = \int_{a_y}^{b_y} \int_{x-(y)}^{x+(y)} f(x, y) \, dx \, dy. \quad (5.26)$$

Since both the integral in Eqn. (5.23) and the integral in Eqn. (5.26) are equal to $I$, we obtain the important result that

$$\int_{a_x}^{b_x} \int_{y-(x)}^{y+(x)} f(x, y) \, dy \, dx = \int_{a_y}^{b_y} \int_{x-(y)}^{x+(y)} f(x, y) \, dx \, dy. \quad (5.27)$$

The only difference between the above two iterated integrals are the order of the integrations. Switching from one to the other is referred to as reversing the order of integration. Making this switch is often a useful approach to evaluating a difficult iterated integral. It is sometimes easier to evaluate an iterated integral in one order than it is in the other order.

Reversing the order of integration usually requires using the equations for one set of boundary curves (e.g. the upper and lower curves) to calculate the other set (e.g. the left and right curves). There is one important case, however, when this task is trivial because the limits of integration are constants.

**Example 5.4** Let the region $A$ be the rectangle with axes parallel to the coordinate axes defined by

$$a_x \leq x \leq b_x, \quad (5.28)$$

and

$$a_y \leq y \leq b_y, \quad (5.29)$$

where $a_x, b_x, a_y,$ and $b_y$ are constants. Then, Eqn. (5.27) becomes

$$\int_{a_x}^{b_x} \int_{a_y}^{b_y} f(x, y) \, dy \, dx = \int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x, y) \, dx \, dy. \quad (5.30)$$

When the region $A$ is not a rectangle, the task of reversing the order of integration involves determining the new limits of integration from the original limits. Suppose we are given the integral

$$I = \int_{a_x}^{b_x} \int_{y-(x)}^{y+(x)} f(x, y) \, dy \, dx,$$

and we wish to reverse the order of integration. We can do that by performing the following steps.

1. Identify the region $A$ of integration for the area integral equal to the given iterated integral, and draw it in a rough sketch. Do this by noting that the region is specified by the inequalities

$$a_x \leq x \leq b_x$$
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and

\[ y_-(x) \leq y \leq y_+(x). \]

Make a rough sketch of the curves \( y = y_-(x) \) and \( y = y_+(x) \) for \( x \) ranging from \( a_x \) to \( b_x \). Then, draw a straight, vertical line segment of arbitrary constant \( x \) between \( a_x \) and \( b_x \) that starts at \( y = y_-(x) \) at the bottom and ends at \( y_+(x) \) at the top. This line represents the region of integration of the \( y \) integral for that value of \( x \). Then, mentally, sweep this line from \( x = a_x \) to \( x = b_x \). This process sweeps out the region \( A \). See Figure 5.8.

![Figure 5.8: Determining the region A.](image1)

![Figure 5.9: Determining new inequalities for A.](image2)

2. Write a new set of inequalities for the region \( A \) in the form

\[ a_y \leq y \leq b_y \]

and

\[ x_-(y) \leq x \leq x_+(y). \]

Do this by removing the vertical line segment from the sketch and adding a horizontal line segment with arbitrary constant \( y \) which passes from the left boundary of \( A \) to the right boundary. This represents the path of the \( x \) integral (with constant \( y \)) with left limit \( x = x_-(y) \) and right limit \( x = x_+(y) \). Determine these functions by inverting the equations \( y = y_-(x) \) and \( y = y_+(x) \) by solving for \( x \). Finally, determine \( a_y \) and \( b_y \), by finding respectively the smallest value of \( y_-(x) \) and the largest value of \( y_+(x) \) for \( x \) in the region \( a_x \leq x \leq b_x \). See Figure 5.9.

3. Write the iterated integral with reversed order of integration with these new limits of integration in the form

\[ I = \int_{a_y}^{b_y} \int_{x_-(y)}^{x_+(y)} f(x, y) \, dx \, dy. \]
To reverse the order of the integrations in an iterated integral of the form

\[ I = \int_{a}^{b_{y}} \int_{x_{-}(y)}^{x_{+}(y)} f(x, y) \, dx \, dy, \]

one can follow the above procedure in reverse.

**Sample Problem 5.3** Reverse the order of integration of the iterated integral

\[ I = \int_{0}^{x_{0}} \int_{0}^{x_{2}^{2}} f(x, y) \, dy \, dx. \]

**Solution:**

**Givens:** \( x_{0} \).

First, we sketch the curves \( y = x^{2} \) and \( y = x_{0}^{2} \), and add the vertical line segment at arbitrary \( x \) with \( 0 \leq x \leq x_{0} \) from the bottom curve to the top one, as illustrated in Figure 5.10. This line segment represents the path of the \( y \)

![Figure 5.10: Determining the region \( A \) for Sample Problem 5.3.](image)

integral. Then, we mentally sweep that line segment from \( x = 0 \) to \( x = x_{0} \) to obtain the region \( A \), which is shaded in the figure.

Next, we replace the vertical line segment with a horizontal line segment at arbitrary \( y \) between \( y = 0 \) and \( y = x_{0}^{2} \) passing from \( x = 0 \) to \( x = \sqrt{y} \) (the inverse of \( y = x^{2} \)). (See Figure 5.11.) We see that when this horizontal line moves from \( y = 0 \) to \( y = x_{0}^{2} \), it sweeps out the same shaded area. Hence, \( A \) is specified by the inequalities

\[ 0 \leq y \leq x_{0}^{2} \]

and

\[ 0 \leq x \leq \sqrt{y}. \]
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Therefore, our area integral can be written in the form

\[ I = \int_{x_0}^{x_0} \int_{y_0}^{\sqrt{y}} f(x, y) \, dx \, dy. \]

**Sample Problem 5.4** Evaluate the iterated integral

\[ I = \int_{x_0}^{x_0} \int_{y_0}^{x_2} x e^{y^2} \, dx \, dy. \]

**Solution:**

**Givens:** \( x_0. \)

The function \( e^{y^2} \) does not have an antiderivative that can be expressed in terms of simple functions. Therefore, we are not able to do the \( y \) integral first as indicated in our iterated integral above. Our only hope for evaluating this integral is to reverse the order of integration. This can be done by applying the results of Sample Problem 5.3. The result is

\[ I = \int_{y_0}^{y_0} \int_{x_0}^{x_2} x e^{y^2} \, dx \, dy. \]

Performing the \( x \) integration, we find

\[ I = \int_{x_0}^{x_0} \left[ x^2 \right]_{x_0}^{\sqrt{y}} e^{y^2} \, dy = \int_{y_0}^{y_0} \frac{y}{2} e^{y^2} \, dy. \]

The \( y \) integration can be performed by using the \( u \) substitution \( u = y^2. \) We then have \( du = \frac{du}{dy} \, dy = 2y \, dy \) so that the integral becomes

\[ I = \frac{1}{4} \int_{x_0}^{x_0} e^u \, du = \frac{1}{4} \left[ e^u \right]_{x_0}^{x_0} = \frac{1}{4} \left( e^{y_0^2} - 1 \right). \]

### 5.5 Evaluation Using Polar Coordinates

If the area of integration has circular or radial boundaries as in Sample Problem 5.2, it is often most convenient to express the variable of integration in terms of polar coordinates \( r \) and \( \theta \) as

\[ x = r \cos \theta \quad (5.31) \]

and

\[ y = r \sin \theta \quad (5.32) \]

illustrated in Figure 5.12. In this approach, we choose our tiny areas to have the shape shown in Figure 5.13. In the limit as \( \Delta r \) and \( \Delta \theta \) approach zero, the shape approaches a rectangle with sides of \( \Delta r \) and \( r \, \Delta \theta. \) Hence, the element of area is given by

\[ dA = r \, d\theta \, dr. \quad (5.33) \]
To change the variables of integration of an iterated integral of the form

\[ I = \int_{a_x}^{b_x} \int_{y_-(x)}^{y_+(x)} f(x, y) \, dy \, dx, \]

or

\[ I = \int_{a_y}^{b_y} \int_{x_-(y)}^{x_+(y)} f(x, y) \, dx \, dy. \]

to polar coordinates, we first note that each of these forms represent an area integral of the form

\[ I = \int \int_A f(x, y) \, dA. \]

From the limits of integration, we can recognize the region \( A \) of integration and can make a rough sketch of it in a Cartesian coordinate system. Then, by studying the sketch, we attempt to express \( A \) in terms of inequalities involving \( r \) and \( \theta \) instead of \( x \) and \( y \). If the inequalities are of the form

\[ r_1 \leq r \leq r_2 \]

and

\[ \theta_1(r) \leq \theta \leq \theta_2(r), \]

then, the region is \( \theta \)-simple, and the integral \( I \) can be written

\[ I = \int_{r_1}^{r_2} \int_{\theta_1(r)}^{\theta_2(r)} f(r \cos \theta, r \sin \theta) \, r \, d\theta \, dr. \]  \hspace{1cm} (5.34)

If, instead, the inequalities are of the form

\[ \theta_1 \leq \theta \leq \theta_2 \]
and

\[ r_1(\theta) \leq r \leq r_2(\theta), \]

then, the region is "\( r \)-simple", and the integral \( I \) can be written

\[
I = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) \ r \ dr \ d\theta. \tag{5.35}
\]

Neither of these integrals have any functions of \( x \) or \( y \) included in their integrands. They have all been replaced with functions of \( r \) and \( \theta \).

**Example 5.5** Consider the crosshatched region \( A \) shown in Figure 5.14. It has a shape which is both \( r \)-simple and \( \theta \)-simple. Hence, we can integrate over \( A \) in either order, doing the \( r \) integral first or the \( \theta \) integral first. Since the limits of integration are constants, it is trivial to change the order of integration from one to the other. The result is

\[
I = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} f(r \cos \theta, r \sin \theta) \ r \ d\theta \ dr = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r \cos \theta, r \sin \theta) \ r \ dr \ d\theta.
\]

Let’s use the result to calculate the area \( A \) of the crosshatched region in the figure. Taking \( f(x, y) = 1 \) and choosing to do the \( \theta \) integration first, we have

\[
A = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} r \ d\theta \ dr = (\theta_2 - \theta_1) \int_{r_1}^{r_2} r \ dr = (\theta_2 - \theta_1) \frac{r_2^2 - r_1^2}{2}.
\]

A sample problem using polar coordinates to integrate a nontrivial integrand over a \( \theta \)-simple region is included in the next section.
5.6 Forces Due to Area Charges

One of the reasons we have studied area integrals is that we need to use them whenever we need to calculate the force \( \vec{F}_q \) exerted on a point charge \( q \) by an extended charge that is distributed over a flat area \( A \). We are now in a position to be able to perform such calculations provided that the shape of \( A \) is \( x \)-simple, \( y \)-simple, \( r \)-simple, or \( \theta \)-simple. We again make use of the material developed in Section 3.6.4 Coulomb’s Law for Distributed Charges.

The force is calculated by a charge integral in the form

\[
\vec{F}_q = k q \int_{Q} \frac{\vec{r}}{r^3} \, dQ,
\]

where

\[
\vec{r} = (x_p - x) \hat{i} + (y_p - y) \hat{j} + (z_p - z) \hat{k}
\]

is the vector from the element of integration \( dQ \) located at \((x, y, z)\) to the point charge \( q \) located at the point \((x_p, y_p, z_p)\). Its magnitude is

\[
r = \sqrt{(x_p - x)^2 + (y_p - y)^2 + (z_p - z)^2}.
\]

If the charge is distributed over a flat area \( A \) with surface charge density (charge per unit area) \( \sigma(x, y) \), then the element \( dQ \) of charge integration is related to element \( dA \) of area integration by

\[
dQ = \sigma(x, y) \, dA
\]

(see Sec. 3.5). Consequently, the charge integral can be converted into a area integral, giving

\[
\vec{F}_q = k q \int_{A} \frac{\vec{r}}{r^3} \sigma(x, y) \, dA.
\]

Since the above area integral involves the adding of many vectors, it is necessary to break the vector equation down into its three scalar components in order to be able to perform the integration. The result is

\[
(F_q)_x = k q \int_{A} \frac{x_p - x}{r^3} \sigma(x, y) \, dA,
\]

\[
(F_q)_y = k q \int_{A} \frac{y_p - y}{r^3} \sigma(x, y) \, dA,
\]

\[
(F_q)_z = k q \int_{A} \frac{z_p - z}{r^3} \sigma(x, y) \, dA.
\]

These equations present complete expressions for the force on the point charge \( q \). If we know the charge \( q \), its location, the region \( A \), and the surface charge density \( \sigma(x, y) \), then the evaluation of the three area integrals is all that remains to be done.

\[\text{Recall that } dA \text{ is always positive, which guarantees the } dQ \text{ has the same sign as } \sigma.\]
Sample Problem 5.5 Consider a flat, circular disk of radius \( R \) located in the \( x\)-\( y \) plane with center at the origin. Electrical charge is distributed uniformly over the disk with surface charge density \( \sigma \). Determine the electrical force \( \vec{F} \) exerted by the disk on a point charge \( q \) located on the \( z \) axis at the point \( z = z_p \). (See Figure 5.15)

Solution:
Givens: \( R, \sigma, z = 0, q, x_p = y_p = 0, z_p \).

Since the geometry is circular, we use polar coordinates and set\(^5\) \( x = \rho \cos \theta \), \( y = \rho \sin \theta \), and \( dA = \rho \, d\theta \, d\rho \). Substitution of the polar expressions for \( x \) and \( y \), and the zero values for \( z \), \( x_p \), and \( y_p \) into the above equation for \( r \) gives

\[
r = \sqrt{\rho^2 + z_p^2}.
\]

Starting with the expression above for the \( x \) component of the force, we find with the above substitutions

\[
(F_q)_x = k q \sigma \int_A \frac{-x}{r^3} \, dA = -k q \sigma \int_0^R \int_0^{2\pi} \frac{\rho \cos \theta}{(\rho^2 + z_p^2)^{3/2}} \rho \, d\theta \, d\rho.
\]

Evaluation of the \( \theta \) integral gives zero, which shows that the \( x \) component of the force is zero. Following the same procedure for the \( y \) component of the force

\(^5\)We use \( \rho \) here instead of \( r \) because we are using the symbol \( r \) to mean something else. It is unimportant what we call it since it is a variable of integration which will disappear from the problem once the definite integral is evaluated.
shows that it is zero also. Thus, we have

\[(F_q)_x = (F_q)_y = 0.\]

Only the \(z\) component of the force remains. We have

\[(F_q)_z = kq \int_A \frac{z_p}{r^3} \sigma \, dA = kq \sigma z_p \int_0^R \int_0^{2\pi} \frac{\rho \, d\theta \, d\rho}{(\rho^2 + z_p^2)^{3/2}} = 2\pi kq \sigma z_p \int_0^R \frac{\rho \, d\rho}{(\rho^2 + z_p^2)^{3/2}}.\]

We now make the \(u\) substitution \(u = \rho^2 + z_p^2\) with \(du = \frac{du}{d\rho} \, d\rho = 2\rho \, d\rho\) to obtain

\[(F_q)_z = 2\pi kq \sigma z_p \int_{z_p^2}^{R^2 + z_p^2} u^{-3/2} \, du = 2\pi kq \sigma \left( 1 - \frac{z_p}{\sqrt{R^2 + z_p^2}} \right).\]

Since the only component of the force which is not zero is the \(z\) component, the force vector can be written as

\[\vec{F}_q = 2\pi kq \sigma \left( 1 - \frac{z_p}{\sqrt{R^2 + z_p^2}} \right) \hat{k}.\]
PROBLEMS

1. Write the double integral

\[ I = \int \int_A f(x, y) \, dA \]

as an iterated integral with the \( x \) integration done first, followed by the \( y \) integration, where the region of integration \( A \) is the shaded region in Figure 5.16.

![Figure 5.16: Geometry for Problems 1 and 2.](image)

2. Write the double integral

\[ I = \int \int_A f(x, y) \, dA \]

as an iterated integral with the \( y \) integration done first, followed by the \( x \) integration, where the region of integration \( A \) is the shaded region in Figure 5.16.

3. Evaluate

\[ I = \int_0^a \int_0^L \sqrt{y}x^3 \, dx \, dy. \]

4. Evaluate

\[ I = \int_0^a \int_0^y \sqrt{y}x^3 \, dx \, dy. \]
5. Evaluate

\[ I = \int_0^{\theta_0} \int_0^R r^2 \sin \theta \, dr \, d\theta. \]

6. Evaluate

\[ I = \int_0^{\theta_0} \int_0^{B \cos \theta} r^2 \sin \theta \, dr \, d\theta. \]

(Solution check: \( I = 1.29 \, m^3 \) for \( B = 3 \, m \) and \( \theta_0 = \pi/5 \) radians.)

7. Evaluate

\[ I = \int_0^B \int_y^B \sqrt{B^2 - x^2} \, dx \, dy \]

by reversing the order of integration. (Solution check: \( I = 2.67 \, m^3 \) for \( B = 2 \, m \).)

8. Evaluate

\[ \int_0^R \int_0^{\sqrt{R^2 - y^2}} \sqrt{x^2 + y^2} \, dx \, dy \]

by changing to polar coordinates. (Solution check: \( I = 14.1 \, m^3 \) for \( R = 3 \, m \).)

9. Find a symbolic solution for the area \( A \) of the shaded region shown in Figure 5.17, where the curve is specified in polar coordinates and \( B \) is a given positive constant. (Solution check: \( A = 9.42 \, m^2 \) for \( B = 2 \, m \).)
10. By evaluating a double integral in the form of an iterated integral with respect to \( x \) and \( y \), find a symbolic solution for the area \( A \) bounded by the graphs of \( y = B \sin kx \) at the upper boundary and \( y = B \cos kx \) at the lower boundary between the two intersection points for these curves at \( x = \pi/(4k) \) and \( x = 5\pi/(4k) \). (Hint: Sketch the two curves together in one figure with the same \( x \) and \( y \) axes.) (Solution check: \( A = 8.49 \, \text{m}^2 \) for \( B = 9 \, \text{m} \) and \( k = 3 \, \text{m}^{-1} \).)

11. (Supplemental problem for students who have studied the chapter on electric potential in their physics text.) A thin, flat rectangular sheet of plastic lies on the \( z = 0 \) plane, occupying the region \( 0 \leq x \leq a, 0 \leq y \leq b \), where \( a \) and \( b \) are positive constants. The sheet is charged with area charge density (charge per unit area) of \( \sigma(x, y) = Kxy \), where \( K \) is a given positive constant. There are no other charges present. With \( V = 0 \) at infinity, find an expression for the electric potential \( V(z) \) as a function of \( z \) along the positive \( z \) axis. (Solution check: \( V = 39.5k = 3.55 \times 10^{11} \, \text{volts} \) for \( a = b = 4 \, \text{m}, K = 3 \, \text{C} \, \text{m}^{-4} \), and \( z = 3 \, \text{m} \).)
Chapter 6

Surface Integrals

6.1 Introduction

This chapter is concerned with the integration of a function over a surface area $S$, which need not be flat. The integrals of interest are referred to as surface integrals and can be written in the form\(^1\)

$$ I = \int_S f(x, y, z) \, dS, \quad (6.1) $$

where the $S$ at the bottom of the integral sign indicates the curved or flat surface over which we are integrating. If the surface is flat and lies in the $x$-$y$ plane, it is belongs to the category of the area integrals discussed in the previous chapter. Some authors use the notation

$$ I = \int_A f(x, y, z) \, dA $$

to indicate a general surface integral, but in this book, we reserve that notation for surface integrals over flat surfaces lying in the $x$-$y$ plane. Another notation commonly used is

$$ I = \iint_S f(x, y, z) \, dS, $$

which indicates explicitly that evaluation of the surface integral requires two definite integrals.

The notation in Eqn. (6.1) does not indicate whether the surface is open (like a hemispherical surface) or closed (like a spherical surface). When it is desired to indicate explicitly that the surface is closed, the notation

$$ I = \oint_S f(x, y, z) \, dS $$

is used.

\(^1\)Since the integration is no longer confined to the plane $z = 0$, we allow the integrand to be a function of $z$. 

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6.2 Definition

The surface integral is defined by the same procedure as the one used to define the area integral in Section 5.2 except that we replace the tiny, flat areas $\Delta A_i$ that make up the flat surface $A$ with tiny, possibly curved areas $\Delta S_i$ that make up the possibly curved surface $S$. The defining equation becomes

$$\int_S f(x, y, z) \, dS = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i. \quad (6.2)$$

Because the definitions of surface integrals and area integrals are so similar, they share many common properties. For example, we have

$$\int_S Kg(x, y, z) \, dS = K \int_S g(x, y, z) \, dS, \quad (6.3)$$

$$\int_S [f(x, y, z) + g(x, y, z)] \, dS = \int_S f(x, y, z) \, dS + \int_S g(x, y, z) \, dS, \quad (6.4)$$

and

$$\int_{S_1+S_2} f(x, y, z) \, dS = \int_{S_1} f(x, y, z) \, dS + \int_{S_2} f(x, y, z) \, dS. \quad (6.5)$$

These properties again follow directly from the definition of surface integrals given in Eqn. (6.2).

6.3 Surface Integral of a Constant

A very important special case of surface integrals to keep in mind is the one with integrand $f(x, y, z) = K$ where $K$ is a constant. Factoring the constant outside of the integral sign, we have

$$I = \int_S K \, dS = K \int_S dS.$$

According to the defining equation for area integrals, Eqn. (6.2), the surface integral on the right-hand-side of the above equation is

$$\int_S dS = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta S_i.$$

Since the sum on the right-hand-side of the above equation adds together the areas of all of the tiny sections that make up $S$, the result of the sum must be the total area $A$ of the surface $S$. The result is

$$\int_S dS = A. \quad (6.6)$$
In words, this states that the surface integral over a surface \( S \) with an integrand of one is equal to the area \( A \) of the surface \( S \). Consequently, the final result for our integral of the constant \( K \) is

\[
I = \int_S K \, dS = K A. \tag{6.7}
\]

This is a very simple result, valid for any surface \( S \). Of course, if the area \( A \) of the surface \( S \) is not known, it is still necessary to evaluate the surface integral \( \int_S dS \), and this can still be a formidable task. If the area is already known, however, then no more effort is necessary. For example, if the region \( S \) is a spherical surface of radius \( R \), then we know that \( A \) is the area of a sphere, which is \( 4\pi R^2 \). Hence, the integral \( I \) in this case is simply \( I = 4K\pi R^2 \).

### 6.4 Evaluation Using Double Integrals

As usual, it is not convenient to evaluate area integrals by using the defining equation Eqn. (6.2). Instead, we need to learn how to express the area integral in terms of definite integrals. In preparation for that task, we first need to introduce the concept of partial derivatives.

#### 6.4.1 Partial Derivatives

If \( f(x, y, z) \) is a function of the variables \( x, y, \) and \( z \), then the object \( \frac{\partial f(x,y,z)}{\partial x} \) is called the partial derivative of \( f(x, y, z) \) with respect to \( x \). It is obtained mathematically by differentiating \( f(x, y, z) \) with respect to \( x \) while holding \( y \) and \( z \) constant. The partial derivative of \( f(x, y, z) \) with respect to \( y \) is written as \( \frac{\partial f(x,y,z)}{\partial y} \), and it is defined the same way except that the derivative is taken with respect to \( y \) while \( x \) and \( z \) are held constant. Similarly for the partial derivative with respect to \( z \).

**Example 6.1** Let \( f(x,y,z) = Ax^2y^5 + y\sin kz - g(x) \), where \( g(x) \) is a function of \( x \) only. Then,

\[
\frac{\partial f(x,y,z)}{\partial x} = 2Ax y^5 - \frac{dg(x)}{dx},
\]

\[
\frac{\partial f(x,y,z)}{\partial y} = 5Ax^2y^4 + \sin kz,
\]

and

\[
\frac{\partial f(x,y,z)}{\partial z} = ky\cos kz.
\]

#### 6.4.2 Transformation Into a Double Integral

Recall that we found in Sec. 4.4.2 that, if the curve \( C \) could be expressed by the equation \( y = y(x) \) for \( b_x > a_x \), where \( y(x) \) is a given function of \( x \), then the
line integral of \( f(x, y) \) over \( C \) could be transformed into the definite integral

\[
\int_C f(x, y) \, ds = \int_{a_x}^{b_x} f[x, y(x)] \sqrt{1 + \left[ \frac{dy(x)}{dx} \right]^2} \, dx.
\]  

(6.8)

This transformation is based on the relationship

\[
ds = \sqrt{1 + \left[ \frac{dy(x)}{dx} \right]^2} \, dx
\]

(6.9)

between the element \( ds \) of length along the curve and the corresponding element \( dx \) of length along the \( x \) axis. When a point moves along the curve a distance \( ds \), the \( x \) coordinate of that point moves the distance \( dx \). The distance \( dx \) is said to be the “projection” of \( ds \) onto the \( x \) axis because \( dx \) is the length of the shadow that would be produced on the \( x \) axis if the interval \( ds \) along the curve were blocking the light in a light beam traveling parallel to the \( y \) axis. Another way to describe this relationship between the two lengths is as follows. If vertical lines are drawn from each point in the interval \( ds \) on the curve to the \( x \) axis, the collection of intersections of these lines with the \( x \) axis would mark out an interval \( dx \).

To obtain a corresponding relationship between the element of area \( dS \) on a surface \( S \) and its projection \( dA \) on the \( x-y \) plane, let the surface \( S \) be defined by the equation

\[
z = z(x, y),
\]

(6.10)

where \( z(x, y) \) is a given function of \( x \) and \( y \). Then, \( dS \) and \( dA \) are related by\[ABD05, p. 1150\]

\[
dS = \sqrt{1 + \left[ \frac{\partial z(x, y)}{\partial x} \right]^2 + \left[ \frac{\partial z(x, y)}{\partial y} \right]^2} \, dA.
\]

(6.11)

The statement that \( dA \) is the projection of \( dS \) onto the \( x-y \) plane means that \( dA \) is the collection of the intersections with the \( x-y \) plane made by vertical lines with constant \( x \) and \( y \) passing through all of the points in \( dS \). It is beyond the scope of this book to derive the above equation, but it can be seen easily that Eqn. 6.11 is the natural extension of Eqn. (6.9) to the two-dimensional case by simply comparing the two equations.

If the surface \( S \) has a region that is parallel to the \( x-z \) plane, then the derivative \( \frac{\partial z(x, y)}{\partial x} \) will be infinite in that region, making Eqn. (6.11) useless there. This difficulty can be avoided by using an equation of the form

\[
y = y(x, z)
\]

(6.12)

to define the surface \( S \), or at least a portion of it that includes the troublesome region. Here \( y(x, z) \) is a function of \( x \) and \( z \). In this case, \( dA = dx \, dz \) is the projection of \( dS \) onto the \( x-z \) plane, which is the collection of the intersections
with the $x$-$z$ plane made by horizontal lines with constant $x$ and $z$ passing through all of the points in $dS$. The two elements are related by [ABD05, p. 1150]

$$dS = \sqrt{1 + \left( \frac{\partial y(x, z)}{\partial x} \right)^2 + \left( \frac{\partial y(x, z)}{\partial z} \right)^2} \, dA. \quad (6.13)$$

Similarly if the surface $S$ has a region that is parallel to the $y$-$z$ plane, then the derivative $\frac{\partial z(x, y)}{\partial y}$ will be infinite in that region, again making Eqn. (6.11) useless there. This difficulty can be avoided by using an equation of the form

$$x = x(y, z) \quad (6.14)$$
to define the surface $S$, or at least a portion of it that includes the troublesome region. Here $x(y, z)$ is a function of $y$ and $z$. In this case, $dA = dy \, dz$ is the projection of $dS$ onto the $y$-$z$ plane, which is the collection of the intersections with the $y$-$z$ plane made by horizontal lines with constant $y$ and $z$ passing through all of the points in $dS$. The two elements are related by [ABD05, p. 1150]

$$dS = \sqrt{1 + \left( \frac{\partial x(y, z)}{\partial y} \right)^2 + \left( \frac{\partial x(y, z)}{\partial z} \right)^2} \, dA. \quad (6.15)$$

The surface integral can be transformed into area integrals over areas in the coordinate planes by using the above formulas. In what follows, we confine our attention to surfaces that are described by Eqn. (6.10), and use Eqn. (6.11) to convert the surface integral into an area integral in the $x$-$y$ plane. Conversion of the surface integral into area integrals in the other coordinate planes can be done using the same approach with only minor modifications.

Using Eqns. (6.10) and (6.11), it is easy to transform the surface integral over the surface $S$ into an area integral over the flat area $A$, which is the projection of $S$ onto the $x$-$y$ plane. The result is

$$I = \int \int_A f[x, y, z(x, y)] \sqrt{1 + \left( \frac{\partial z(x, y)}{\partial x} \right)^2 + \left( \frac{\partial z(x, y)}{\partial y} \right)^2} \, dA, \quad (6.16)$$

where we have evaluated the function $f(x, y, z)$ on the surface $S$ by setting $z = z(x, y)$.

We are now ready to use our knowledge of area integrals gained from the previous chapter. For example, suppose that the shape of $A$ is $y$-simple, defined by the inequalities

$$a_x \leq x \leq b_x \quad (6.17)$$

and

$$y_-(x) \leq y \leq y_+(x), \quad (6.18)$$

where $a_x$ and $b_x$ are constants that express the extreme values of $x$ within $A$, $y = y_+(x)$ is the equation for the upper boundary of $A$, and $y = y_-(x)$ is
the equation for the lower boundary of \( A \). Then, our surface integral can be expressed in terms of the iterated integral

\[
I = \int_{a_x}^{b_x} \int_{y_-(x)}^{y_+(x)} f[x, y, z(x, y)] \sqrt{1 + \left[ \frac{\partial z(x, y)}{\partial x} \right]^2 + \left[ \frac{\partial z(x, y)}{\partial y} \right]^2} \, dy \, dx. \tag{6.19}
\]

On the other hand, suppose the shape of \( A \) is \( x \)-simple, specified by the inequalities

\[
a_y \leq y \leq b_y \tag{6.20}
\]

and

\[
x_-(y) \leq x \leq x_+(y), \tag{6.21}
\]

where \( a_y \) and \( b_y \) are constants that specify the extreme values of \( y \) within \( A \), \( x = x_+(y) \) is the equation for the right boundary of \( A \) and \( x = x_-(y) \) is the equation for the left boundary of \( A \). Then, our surface integral can be expressed in terms of the iterated integral

\[
I = \int_{a_y}^{b_y} \int_{x_-(y)}^{x_+(y)} f[x, y, z(x, y)] \sqrt{1 + \left[ \frac{\partial z(x, y)}{\partial x} \right]^2 + \left[ \frac{\partial z(x, y)}{\partial y} \right]^2} \, dx \, dy. \tag{6.22}
\]

Another possibility is that \( A \) is \( \theta \)-simple, specified by the inequalities

\[
r_1 \leq r \leq r_2
\]

and

\[
\theta_1(r) \leq \theta \leq \theta_2(r)
\]

in polar coordinates. Then, our surface integral can be written as

\[
I = \int_{\theta_1(r)}^{\theta_2(r)} \int_{r_1}^{r_2} f[x, y, z(x, y)] \sqrt{1 + \left[ \frac{\partial z(x, y)}{\partial x} \right]^2 + \left[ \frac{\partial z(x, y)}{\partial y} \right]^2} \, r \, dr \, d\theta. \tag{6.23}
\]

In the above equation, we need to make substitutions \( x = r \cos \theta \) and \( y = r \sin \theta \) in both \( f[x, y, z(x, y)] \) and in the functions in the square root before the integration is performed. In the case of the square root, however, the substitutions should not be made until after the partial derivatives are taken.

The final possibility is that \( A \) is \( r \)-simple, specified by

\[
\theta_1 \leq \theta \leq \theta_2
\]

and

\[
r_1(\theta) \leq r \leq r_2(\theta)
\]

in polar coordinates. Then, our surface integral can be written as

\[
I = \int_{\theta_1(\theta)}^{\theta_2(\theta)} \int_{r_1(\theta)}^{r_2(\theta)} f[x, y, z(x, y)] \sqrt{1 + \left[ \frac{\partial z(x, y)}{\partial x} \right]^2 + \left[ \frac{\partial z(x, y)}{\partial y} \right]^2} \, r \, dr \, d\theta. \tag{6.24}
\]
6.4. EVALUATION USING DOUBLE INTEGRALS

Again, we need to make substitutions $x = r \cos \theta$ and $y = r \sin \theta$ in both $f[x, y, z(x, y)]$ and in the functions in the square root before the integration is performed. In the case of the square root, the substitutions should not be made until after the partial derivatives are taken.

**Sample Problem 6.1** Consider a circular cylinder of radius $R$ located with its axis lying along the $x$ axis. The surface of the cylinder is charged with surface charge density $\sigma$ (charge per unit area) which varies with height above the $x$-$y$ plane as $\sigma = Bz$, where $B$ is a constant. Determine the total charge $Q$ located on the portion of the cylinder that lies from $x = 0$ to $x = L$ in the first octant. (See Figure 6.1)

**Solution:**

**Givens:** $R, B, L$.

We have

\[
Q = \int_S \sigma \, dS = B \int_S z \, dS,
\]

where the surface $S$ is given by

\[
0 \leq x \leq L,
\]

\[
0 \leq y \leq R,
\]

\[
z = \sqrt{R^2 - y^2}.
\]

Evaluation of the integrand on the surface $S$ gives

\[
Q = B \int_S \sqrt{R^2 - y^2} \, dS.
\]
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Since \( \frac{\partial z}{\partial x} = 0 \) and \( \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - y^2}} \) on the surface, we have

\[
dS = \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dA = \sqrt{1 + \frac{y^2}{R^2 - y^2}} \, dA = \frac{R}{\sqrt{R^2 - y^2}} \, dA.
\]

Substitution of this result into our surface integral gives

\[
Q = B \int_A \sqrt{R^2 - y^2} \frac{R}{\sqrt{R^2 - y^2}} \, dA = BR \int_A \, dA = BRA,
\]

where \( A \) is the area of the region \( A \). As can be seen from Figure 6.1, the projection \( A \) of the surface \( S \) onto the \( x-y \) plane is just the rectangle \( 0 \leq x \leq L \) and \( 0 \leq y \leq R \), which has area \( A = RL \). Hence, the result for \( Q \) becomes

\[
Q = BR^2L.
\]

Sample Problem 6.2 Determine the surface area \( A \) of the paraboloid \( z = B(x^2 + y^2) \) between the planes \( z = 0 \) and \( z = z_0 \), where \( B \) and \( z_0 \) are constants. (See Figure 6.2)

Solution:

Givens: \( B, z_0 \).

We need to evaluate the surface integral

\[
A = \int_S dS,
\]

where \( S \) is given by \( z = B(x^2 + y^2) \) with \( z \) ranging from 0 to \( z_0 \). Since \( \frac{\partial z}{\partial x} = 2Bx \) and \( \frac{\partial z}{\partial y} = 2By \) on the surface, we have

\[
dS = \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dA = \sqrt{1 + 4B^2x^2 + 4B^2y^2} \, dA = \sqrt{1 + 4B^2(x^2 + y^2)} \, dA.
\]

Consequently, our surface integral can be written as the area integral

\[
A = \int_A \sqrt{1 + 4B^2(x^2 + y^2)} \, dA,
\]

where \( A \) is the projection of \( S \) onto the \( x-y \) plane. To determine what that projection is, we note that the surface \( S \) intersects any plane of constant \( z \) with a circle of radius \( \sqrt{x^2 + y^2} = \sqrt{z/B} \). The largest of these circles is the one at

---

\[2\] This solution glosses over a potentially troublesome point, which is that the expression for \( dS \) in terms of \( dA \) includes a factor \((R^2 - y^2)^{-1/2}\) which is infinite at the point \( y = R \). This difficulty could be handled by taking the region of the \( y \) integration to be from \( y = 0 \) to \( y = a \) with \( a < R \) and then taking the limit as \( a \) tends to \( R \) after doing the integration. In this sample problem, however, that process is not required because the factor \( z = (R^2 - y^2)^{1/2} \) included in the integrand removes the singularity.
the top at \( z = z_0 \). Consequently, as \( z \) varies from 0 to \( z_0 \), the values of \( x^2 + y^2 \) vary from 0 to \( z_0 / B \). This means that the projection \( A \) of \( S \) on the \( x-y \) plane is the circular disk of radius \( \sqrt{z_0 / B} \). Because the geometry is circular, we apply polar coordinates. The result is

\[
A = \int_0^{\sqrt{z_0 / B}} \int_0^{2\pi} \sqrt{1 + 4B^2r^2} r \, d\theta \, dr = 2\pi \int_0^{\sqrt{z_0 / B}} \sqrt{1 + 4B^2r^2} r \, dr.
\]

The \( r \) integral can be evaluated by making the \( u \) substitution \( u = 1 + 4B^2r^2 \) with \( du = 8B^2r \, dr \) to obtain

\[
A = \frac{2\pi}{8B^2} \int_1^{1+4Bz_0} u^{1/2} \, du = \frac{\pi}{4B^2} \left[ \frac{u^{3/2}}{3/2} \right]_{1}^{1+4Bz_0} = \frac{\pi}{6B^2} \left[ (1 + 4Bz_0)^{3/2} - 1 \right].
\]

### 6.5 Flux Integrals

#### 6.5.1 Introduction

We now turn our attention to a special class of surface integrals which are especially important in the subject of electricity and magnetism. In fact, flux integrals occur in the integral form of all four of Maxwell’s equations.

These integrals involve the concept of “vector fields”. A vector field is a function that associates with each point in a region of space a unique vector \( \vec{V}(x, y, z) \). Examples include the velocity of a fluid throughout the fluid, the gravitational field in space, the electric field in space, and the magnetic field in space.

The class of integrals known as flux integrals gets its name from the integral used to calculate the rate of flow of a fluid through a surface. If the fluid has mass density (mass per unit volume) \( \rho(x, y, z) \) and velocity \( \vec{v}(x, y, z) \), then the rate of fluid flow (mass per unit time) \( R \) through a surface \( S \) is given by the surface integral

\[
R = \int_S \vec{V} \cdot \hat{n} \, dS,
\]

where \( \vec{V} \) is a vector field defined by \( \vec{V} = \rho(x, y, z)\vec{v}(x, y, z) \), and \( \hat{n} \) is a unit vector that is normal to the surface \( S \) at the location of \( dS \). The integrand is positive when the fluid passes through the surface in the same direction as \( \hat{n} \) and is negative when it flows in the opposite direction. The integral is said to express the flux of the fluid through or across the surface.

The term “flux” has become more generalized to mean the integral of any vector field \( \vec{V} \) in the form of Eqn. (6.25) even if \( \vec{V} \) does not represent the flow of anything. For example, we speak of the flux \( \Phi_E \) of the electric field \( \vec{E} \) through a surface even if the electric field is static and nothing is flowing. Usually, when a flux integral is written, we combine the unit vector \( \hat{n} \) with the element \( dS \) of surface area to define a new vector

\[
\vec{dS} = \hat{n} \, dS
\]
and write the flux \( \Phi_V \) of any vector field \( \vec{V} \) over the surface \( S \) as

\[
\Phi_V = \int_S \vec{V} \cdot d\vec{S}. 
\]

(6.26)

Since the surface \( S \) has two sides\(^3\), there are two (antiparallel) normal vectors at each point on the surface. It is necessary to specify which of these is to be used when specifying the above integral. This can be done by stating that \( d\vec{S} \) has a positive \( z \) component, for example, in which case, we say we are using upward normals. There is one situation, however, when no statement is necessary because there is a standard convention. If the surface \( S \) is closed (it has an inside and an outside), then the standard choice, which need not be specified, is that the \( d\vec{S} \) points outward from the interior of the surface.

Even though flux integrals have vectors in their integral, they still are surface integrals of a scalar, and therefore, belong to the same class of integrals that we have been discussing previously in this chapter. To display the scalar explicitly, recall that the scalar product of two vectors can be written

\[
\vec{A} \cdot \vec{B} = AB \cos \phi, 
\]

(6.27)

where \( A \) and \( B \) are the magnitudes of the corresponding vectors and \( \phi \) is the angle between the two vectors when placed tail to tail. This means that our flux integral can be written

\[
\Phi_V = \int_S V \cos \phi \ dS, 
\]

(6.28)

where \( V \) is the magnitude of \( \vec{V} \) at the location of \( dS \), and \( \phi \) is the angle between \( \vec{V} \) and \( \hat{n} \) at the same place. This shows that the flux integral is just an ordinary surface integral of a scalar as discussed previously.

These surface integrals do have a feature, however, which we have not previously encountered. The integrand depends on the angle between a vector and the normal to the surface at the location of the element of integration. Previously, it was implied that the integrand \( f(x, y, z) \) was a function defined independently of the surface \( S \). Here, that function is definitely influenced by the surface. If we change the surface without changing \( \vec{V} \), the scalar integrand is changed.

In particular, it is interesting to note what happens if we change our choice of the unit normal to the surface so that it points in the opposite direction. The only change this makes in our scalar integrand is that it adds 180 degrees to the angle \( \phi \) in the integrand. This simply changes the sign of the integrand at every point on the surface, which, in turn, changes the sign of resulting integral.\(^4\)

\( ^3 \)We are excluding from consideration pathological surfaces with only one side such as the infamous Mobius strip.

\( ^4 \)Note, also, that the same change in sign of the integral occurs for the same reason if we reverse the direction of the vector \( \vec{V} \) instead of reversing the direction of the unit normal.
6.5.2 Simple Special Cases

Simple Surfaces

Flux integrals simplify greatly when the surface $S$ is a plane that is perpendicular to one of the coordinate axes. Suppose, for example, the surface $S$ is a portion of the plane $x = K$ with its unit normal oriented along the positive $x$ direction, where $K$ is a constant. Then, the unit vector of the surface is simply

$$\hat{n} = \hat{i}$$

at every point on the surface. Now, let the portion of the plane that comprises the surface $S$ be the simple area $A$. Then, the magnitude of the vector $d\vec{S}$ can be written as $dA$, and the vector itself can be written as

$$d\vec{S} = \hat{i} \, dA.$$  

Hence, it follows from the vector identity

$$\vec{C} \cdot \vec{B} = C_x B_x + C_y B_y + C_z B_z,$$

that flux integral can be written as

$$\int_S \vec{V} \cdot d\vec{S} = \int_A V_x(K, y, z) \, dA,$$

where $V_x(K, y, z)$ is the $x$ component of the vector $V$ at the location $(K, y, z)$ of $dS$ on the surface. If the region $A$ is $y$-simple, we can express this integral as an iterated integral by setting $dA = dy \, dz$ and integrating both the $y$ and $z$ integrals in the positive directions. Or, if the region $A$ is $z$-simple, we can do the same by setting $dA = dz \, dy$.

Simple Integrand

A very important special case, which we will encounter frequently in our studies, is when the magnitude of the vector $\vec{V}$ and the angle between that vector and the normal vector are both constant over the entire surface. Then, our flux integral in Eqn. (6.28) becomes

$$\Phi_V = V \cos \phi \int_S dS = V \cos \phi A,$$

(6.29)

where $A$ is the surface area of $S$. In particular, $\Phi_V = V_A$ when $\vec{V}$ is parallel to $d\vec{S}$ everywhere on the surface, and $\Phi_V = -V_A$ when $\vec{V}$ is antiparallel to $d\vec{S}$ everywhere on the surface.

Sample Problem 6.3 Determine the flux of the vector $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$ through a spherical surface of radius $R$ centered at the origin.
CHAPTER 6. SURFACE INTEGRALS

Solution:
Givens: \( R \).

Recall that the vector from \((x_p, y_p, z_p)\) to \((x, y, z)\) is
\[
\vec{r} = (x - x_p)i + (y - y_p)j + (z - z_p)k.
\]
This means that our given vector \( \vec{V} \) is the vector from the origin to the point \((x, y, z)\), which shows that it is parallel to the outward normal of our given sphere at every point on the sphere. Its magnitude is
\[
V = \sqrt{V_x^2 + V_y^2 + V_z^2} = \sqrt{x^2 + y^2 + z^2} = R \text{ everywhere on the sphere.}
\]
Hence, our flux integral is
\[
\Phi_V = V \cdot A = RA = R(4\pi R^2) = 4\pi R^3.
\]

6.5.3 The Unit Normal to the Surface

In order to evaluate flux integrals which are not so simple as those treated in the previous section, we need to be able to express the vector \( \hat{n} \) mathematically in terms of the equation for the surface. Suppose the surface is specified by the equation
\[
z = z(x, y), \tag{6.30}
\]
where \( z(x, y) \) is a function given for \( x \) and \( y \) in a region \( A \) which is the projection of \( S \) onto the \( x-y \) plane. It is beyond the scope of this book to derive the result, but it can be shown that the unit normal to the surface with positive \( z \) component is given as a function of \( x \) and \( y \) by [ABD05, p.1160]
\[
\hat{n} = -\frac{\partial z(x, y)}{\partial x}i + \frac{\partial z(x, y)}{\partial y}j + k \sqrt{1 + \left[\frac{\partial z(x, y)}{\partial x}\right]^2 + \left[\frac{\partial z(x, y)}{\partial y}\right]^2}. \tag{6.31}
\]
The corresponding equation for the unit normal with negative \( z \) component has the right-hand side of this expression replaced by its negative.

As was pointed out in Section 6.4.2, the partial derivatives in the above expression can become infinite on portions of the surface \( S \) that are parallel to the coordinate planes. For this reason, we need also to know the expressions for the unit normal vector when the surface is specified by other functions. In particular, if the surface \( S \) is defined by
\[
y = y(z, x), \tag{6.32}
\]
then, the unit normal vector is given by [ABD05, p.1160]
\[
\hat{n} = -\frac{\partial y(z, x)}{\partial x}i + \frac{\partial y(z, x)}{\partial z}j - \frac{\partial y(z, x)}{\partial z}k \sqrt{1 + \left[\frac{\partial y(z, x)}{\partial x}\right]^2 + \left[\frac{\partial y(z, x)}{\partial z}\right]^2}. \tag{6.33}
\]
The corresponding equation for the unit normal with negative $y$ component has the right-hand side of this expression replaced by its negative.

Similarly, if the surface $S$ is defined by
\[ x = x(y, z), \] (6.34)
then, the unit normal vector is given by\[ \hat{n} = \mathbf{i} - \frac{i\partial x(y, z)}{\partial y} \mathbf{j} - \frac{i\partial x(y, z)}{\partial z} \mathbf{k}, \] (6.35)

The corresponding equation for the unit normal with negative $x$ component has the right-hand side of this expression replaced by its negative.

### 6.5.4 Evaluation By Double Integrals

We are now ready to transform our flux integral into an area integral in one of the coordinate planes. In what follows, we confine our attention to surfaces that are described by Eqn. (6.30)
\[ z = z(x, y), \]
and convert the flux integral into an area integral in the $x$-$y$ plane. Conversion of the flux integral into area integrals in the other coordinate planes can be done using the same approach with only minor modifications.

Another way to write the scalar product between the vectors $\mathbf{A}$ and $\mathbf{B}$ is to express them in terms of their components. We have
\[ \mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \]

Using this equation and Eqn. (6.31) to determine the scalar components of the unit normal vector, we can now write the expression for the scalar product in the flux integral in terms of the components of the vectors. The result is
\[ \mathbf{V} \cdot d\mathbf{S} = \mathbf{V} \cdot \hat{n} dS = \frac{-\partial z(x, y)}{\partial x} V_x - \frac{\partial z(x, y)}{\partial y} V_y + V_z}{\sqrt{1 + \left( \frac{\partial z(x, y)}{\partial x} \right)^2 + \left( \frac{\partial z(x, y)}{\partial y} \right)^2}} dS \] (6.36)

To convert this surface integral into an area integral in the $x$-$y$ plane, we need to express $dS$ in terms of its projection $dA$ in the $x$-$y$ plane using Eqn. (6.11)
\[ dS = \sqrt{1 + \left( \frac{\partial z(x, y)}{\partial x} \right)^2 + \left( \frac{\partial z(x, y)}{\partial y} \right)^2} dA. \]

Comparing the two equations above, we see that they both involve the same square root, which are going to cancel each other when the latter is substituted.
into the former. The result gives for the flux integral
\[ \Phi_V = \int_S \vec{V} \cdot d\vec{S} = \int_A \left[ -\frac{\partial z(x,y)}{\partial x} V_x[x,y,z(x,y)] - \frac{\partial z(x,y)}{\partial y} V_y[x,y,z(x,y)] + V_z[x,y,z(x,y)] \right] dA, \]
valid when the normal has positive \( z \) component. Multiply the right-hand side of the equation by -1 if the normal has negative \( z \) component.

The area integral in Eqn. (6.37) is ready to be written as iterated integrals if the area \( A \) has an appropriate shape. For unit normal vectors with positive \( z \) components, we have
\[ \Phi_V = \int_{y_-(x)}^{y_+(x)} \int_{x_-(y)}^{x_+(y)} \left[ -\frac{\partial z(x,y)}{\partial x} V_x[x,y,z(x,y)] - \frac{\partial z(x,y)}{\partial y} V_y[x,y,z(x,y)] + V_z[x,y,z(x,y)] \right] dy \, dx \]
for \( y \)-simple shapes,
\[ \Phi_V = \int_{x_-(y)}^{x_+(y)} \int_{y_-}^{y_+} \left[ -\frac{\partial z(x,y)}{\partial x} V_x[x,y,z(x,y)] - \frac{\partial z(x,y)}{\partial y} V_y[x,y,z(x,y)] + V_z[x,y,z(x,y)] \right] dx \, dy \]
for \( x \)-simple shapes,
\[ \Phi_V = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} \left[ -\frac{\partial z(x,y)}{\partial x} V_x[x,y,z(x,y)] - \frac{\partial z(x,y)}{\partial y} V_y[x,y,z(x,y)] + V_z[x,y,z(x,y)] \right] r \, d\theta \, dr \]
for \( \theta \)-simple shapes, and
\[ \Phi_V = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \left[ -\frac{\partial z(x,y)}{\partial x} V_x[x,y,z(x,y)] - \frac{\partial z(x,y)}{\partial y} V_y[x,y,z(x,y)] + V_z[x,y,z(x,y)] \right] r \, dr \, d\theta \]
for \( r \)-simple shapes. Polar coordinates \( x = r \cos \theta \) and \( y = r \sin \theta \) need to be substituted in for \( x \) and \( y \) in the square brackets in the last two equations after the differentiation is taken but before the integration is performed. The right-hand sides of all of these equations needs to be multiplied by -1 if the surface normal has negative \( z \) component.

When applying these equations, keep in mind that the element of area \( dA \) is always positive. This means that \( dx, dy, dr, \) and \( d\theta \) should all be positive, requiring us to integrate in the positive direction in all of the above iterated integrals. This requires that the upper limit of integration must always be larger than the lower limit.

**Sample Problem 6.4** Determine the flux of the vector \( \vec{V}(x,y,z) = az \hat{i} + b \hat{j} + y \hat{k} \) through the portion of the plane surface \( x + y + z = c \) that is located in the first octant as illustrated in Figure 6.3. The quantities \( a, b, \) and \( c \) are constants, and \( c \) is positive.

**Solution:**
**Givens:** \( a, b, c. \)
As can be seen from the figure, the projection $A$ of $S$ onto the $x$-$y$ plane is a triangle with one side on the $x$ axis from 0 to $c$ and another side on the $y$ axis from 0 to $c$. An equation for the third side is $y = c - x$. Therefore, the region $A$ is defined by the inequalities

$$0 \leq x \leq c$$

and

$$0 \leq y \leq c - x.$$ 

This shows that the shape of $A$ is $y$-simple, and we can apply Eqn. (6.38),

$$
\Phi_V = \int_0^c \int_0^{c-x} \left[ -\frac{\partial z(x, y)}{\partial x} V_x[x, y, z(x, y)] - \frac{\partial z(x, y)}{\partial y} V_y[x, y, z(x, y)] + V_z[x, y, z(x, y)] \right] dy \, dx.
$$

Since the equation for the surface $S$ can be taken to be

$$z = z(x, y) = c - x - y,$$

we have

$$\frac{\partial z(x, y)}{\partial x} = \frac{\partial z(x, y)}{\partial y} = -1.$$ 

Moreover, since our vector of interest is $\vec{V}(x, y, z) = az\hat{i} + b\hat{j} + y\hat{k}$, we have

$$V_x[x, y, z(x, y)] = az(x, y) = a(c - x - y),$$

$$V_y[x, y, z(x, y)] = b,$$

and

$$V_z[x, y, z(x, y)] = y.$$
Substitution of these five quantities into our flux integral gives
\[ \Phi_V = \int_0^c \int_0^{c-x} \left[ a(c-x-y) + b + y \right] dy \, dx. \]

Collecting terms with the same power of \( y \),
\[ \Phi_V = \int_0^c \int_0^{c-x} \left[ (ac-ax+b) + (1-a)y \right] dy \, dx, \]
and doing the \( y \) integration,
\[ \Phi_V = \int_0^c \left[ (ac-ax+b)y + (1-a)\frac{y^2}{2} \right]_0^{c-x} dx, \]
yields
\[ \Phi_V = \int_0^c \left[ (ac-ax+b)(c-x) + (1-a)\frac{(c-x)^2}{2} \right] dx. \]

Next, we expand the squared term at the far right and collect terms with the same power of \( x \),
\[ \Phi_V = \int_0^c \left[ \left( \frac{1}{2}ac^2 + bc + \frac{1}{2}c^2 \right) - (ac+b+c)x + \frac{1}{2}(1+a)x^2 \right] dx, \]
and do the \( x \) integration
\[ \Phi_V = \left[ \left( \frac{1}{2}ac^2 + bc + \frac{1}{2}c^2 \right) x - (ac+b+c)\frac{x^2}{2} + \frac{1}{2}(1+a)\frac{x^3}{3} \right]_0^c, \]
to obtain
\[ \Phi_V = \left( \frac{1}{2}ac^2 + bc + \frac{1}{2}c^2 \right) c - (ac+b+c)\frac{c^2}{2} + \frac{1}{2}(1+a)\frac{c^3}{3} = \frac{1}{2}bc^2 + \frac{1}{6}(1+a)c^3. \]

**Sample Problem 6.5** Apply Eqn. (6.40) to determine the flux of the vector \( \vec{V}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k} \) through a spherical surface of radius \( R \) centered at the origin.

**Solution:**
\textbf{Givens: } \( R \).

We have already determined the required flux in Sample Problem 6.3, but we did it in a different way. This time we use the equation
\[ \Phi_V = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} \left[ -\frac{\partial z(x, y)}{\partial x} V_x[x, y, z(x, y)] - \frac{\partial z(x, y)}{\partial y} V_y[x, y, z(x, y)] + V_z[x, y, z(x, y)] \right] r \, d\theta \, dr. \]

The equation for the top half of the sphere is
\[ z = z(x, y) = \sqrt{R^2 - x^2 - y^2} \]
6.5. FLUX INTEGRALS

and for the bottom half is

\[ z = z(x, y) = -\sqrt{R^2 - x^2 - y^2}. \]

Consequently, we need to treat the two hemispheres separately. From the symmetry of the problem, however, it is clear that the fluxes through the different hemispheres are equal to each other. Therefore, we need find only the flux through the top half of the sphere and then double the result to obtain the total flux through the sphere.

As can be seen from Figure 6.4, the upper hemisphere projects down onto the \( x-y \) plane as the circular disk of radius \( R \) centered at the origin. Therefore, the projection \( A \) of the hemispherical surface is defined by the inequalities \( 0 \leq r \leq R \) and \( 0 \leq \theta \leq 2\pi \). Hence, our expression for the required flux becomes

\[
\Phi_V = 2 \int_0^R \int_0^{2\pi} \left[ -\frac{\partial z(x, y)}{\partial x} V_x[x, y, z(x, y)] - \frac{\partial z(x, y)}{\partial y} V_y[x, y, z(x, y)] + V_z[x, y, z(x, y)] \right] r \, d\theta \, dr.
\]

We need to evaluate the derivatives in the integrand before we convert them to polar coordinates. Writing our equation for the surface as \( z = z(x, y) = (R^2 - x^2 - y^2)^{1/2} \), we find by using the chain rule for differentiation

\[
\frac{\partial z(x, y)}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}
\]

and

\[
\frac{\partial z(x, y)}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}.
\]

Substituting these expressions into the integrand, along with the expressions for the components of the vector \( \mathbf{V} \) given by

\[
V_x[x, y, z(x, y)] = x,
\]

\[
V_y[x, y, z(x, y)] = y,
\]

and

\[
V_z[x, y, z(x, y)] = z(x, y) = \sqrt{R^2 - x^2 - y^2},
\]

we have

\[
\Phi_V = 2 \int_0^R \int_0^{2\pi} \left[ \frac{x^2}{\sqrt{R^2 - x^2 - y^2}} + \frac{y^2}{\sqrt{R^2 - x^2 - y^2}} + \frac{R^2}{\sqrt{R^2 - x^2 - y^2}} \right] r \, d\theta \, dr.
\]

Putting the three terms over a common denominator, the result is

\[
\Phi_V = 2 \int_0^R \int_0^{2\pi} \frac{R^2}{\sqrt{R^2 - x^2 - y^2}} r \, d\theta \, dr.
\]
Now, we are ready to transform the integrand to polar coordinates by setting $x = r \cos \theta$ and $y = r \sin \theta$ to obtain

$$\Phi_V = 2R^2 \int_0^R \int_0^{2\pi} \frac{r \, d\theta \, dr}{\sqrt{R^2 - r^2}} = 4\pi R^2 \int_0^R \frac{r \, dr}{\sqrt{R^2 - r^2}}$$

where we have performed the $\theta$ integration in the last step. To perform the final integration, we make the $u$ substitution $u = R^2 - r^2$ with $du = -2r \, dr$ to obtain

$$\Phi_V = 4\pi R^2 \int_{R^2}^0 u^{-1/2} (-\frac{1}{2} \, du) = 2\pi R^2 \left[ \frac{u^{1/2}}{1/2} \right]_0^{R^2} = 4\pi R^3.$$

This is the same answer as the one we obtained with much less effort in Sample Problem 6.3.
6.5. FLUX INTEGRALS

PROBLEMS

1. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = \frac{xy}{x-y}$. (Solution check: The numerical values with $x = 2 \text{ m}$ and $y = -2 \text{ m}$ are respectively -1/4 and 1/4.)

2. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = \frac{4xy}{\sqrt{x^2+y^2}}$. (Solution check: The numerical values with $x = 4 \text{ m}$ and $y = 3 \text{ m}$ are respectively .864 and 2.05.)

3. Evaluate $I = \int \int_S (x - 2y - z) \, dS$ for the plane surface $S$ given by $z = B - Cx - Dy$ for $0 \leq x \leq a, 0 \leq y \leq b$ where $B, C, D, a,$ and $b$ are positive constants. (Solution check: The numerical value with $B = 1 \text{ m}, C = 2, D = 3, a = 4 \text{ m},$ and $b = 5 \text{ m}$ is 561 $m^3$.)

4. Find the surface area of the portion of the surface $z = Bxy$ that is inside the cylinder $x^2 + y^2 = R^2$, where $B$ and $R$ are positive constants. (Solution check: The numerical value with $B = 2 \text{ m}^{-1}$ and $R = 3 \text{ m}$ is 117 $m^2$.)

5. Find the surface area of the portion of the paraboloid $z = a(b^2 - x^2 - y^2)$ that lies above the $x$-$y$ plane, where $a$ and $b$ are positive constants. (Solution check: The numerical value with $a = 3 \text{ m}^{-1}$ and $b = 4 \text{ m}$ is $257\pi \text{ m}^2$.)

6. Evaluate $I = \int \int_S \frac{\partial f}{\partial x} \, dS$ with $S$ given by $z = B(x^2 + y^2)$ for $a \leq x^2 + y^2 \leq b$, where $B, a,$ and $b$ are positive constants. (Solution check: The numerical value with $B = 1 \text{ m}^{-1}, a = 4 \text{ m}^2,$ and $b = 16 \text{ m}^2$ is 0.)

7. Use Eqn. (6.28) to find the flux of the vector $\vec{V} = B\hat{i}$ through the portion of the plane $x = a$ (with normal in the positive $x$ direction) that includes the points $0 \leq y \leq b_y, 0 \leq z \leq b_z$, where $B, a, b_y,$ and $b_z$ are positive constants.

8. Find the flux of the vector $\vec{V} = x^2y^3z^4\hat{i} + y^2z^3x^4\hat{j} + z^2x^3y^4\hat{k}$ through the same surface as in the problem just above. You don’t need to use Eqn. (6.28) in this problem. (Solution check: The numerical value with $a = 4 \text{ m}, b_y = 3 \text{ m},$ and $b_z = 2 \text{ m}$ is $2.07 \times 10^8 \text{ m}^{11}$.)

9. Consider a circular cylinder of radius $R$ located with its axis lying along the $x$ axis. Let the surface $S$ be the portion of that cylinder in the first octant that lies from $x = 0$ to $x = L$ as shown in Figure 6.5. Take the normal to the surface to be outward from the $x$ axis. Note that the surface $S$ includes only the curved surface and does not include any planar surface. Find the flux of the vector $\vec{V} = xi + yj + zk$ through the surface $S$. (Solution check: The numerical value with $R = 5 \text{ m}$ and $L = 4 \text{ m}$ is $157 \text{ m}^3$.)

10. Find the flux of the vector $\vec{V} = xi + yj + Bjk$ through the portion of the cone (with upward normal) given by $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = a$ and $z = b$, where $B$ is a positive constant and $a$ and $b$ are
constants that satisfy $0 \leq a < b$. (Solution check: The numerical value with $B = 2$, $a = 1 \text{ m}$, and $b = 2 \text{ m}$ is $14.7 \text{ m}^3$.)

11. (Supplemental problem for students who have studied the chapter on Gauss’ law in their physics text.) Let the electric field be given by $\vec{E}(x, y, z) = Bz \hat{k}$ throughout the region of interest, where $B$ is a positive constant. Determine the net charge within the paraboloid $z = a(b^2 - x^2 - y^2)$ above the plane $z = 0$, where $a$ and $b$ are positive constants. (Solution check: The numerical value with $B = 3 \ N/(C \text{ m})$, $a = 2 \text{ m}^{-1}$, and $b = 4 \text{ m}$ is $2.14 \times 10^{-8} \text{ C}$.)

12. (Supplemental problem for students who have studied the chapter on induction in their physics text.) Consider a magnetic field given by $\vec{B} = K(x^3z^2\hat{i} - x^2z^3\hat{k}) \sin \omega t$ in the region of interest, where $K$ and $\omega$ are positive constants and $t$ is variable time. Show that the magnitude of the induced emf around a circle of radius $R$ in the plane $z = a$ with its center at $x = 0, y = 0, z = a$ (where $R$ and $a$ are positive constants) is $E = \frac{4}{3} \pi a^3 R^4 \omega \cos \omega t$. 

Figure 6.5: Geometry for Problem 9.
Chapter 7

Line Integrals Involving Vectors

7.1 Introduction

In our study of forces exerted on a point charge $q$ by charge distributed along a curved line in Sec. 4.6, we encountered line integrals of the form

$$\vec{F}_q = kq \int_C \frac{\vec{r}}{r^3} \lambda(s) \, ds.$$ 

These integrals belong to a general class of integrals of the form

$$\vec{I} = \int_C \vec{V}(x, y, z) \, ds,$$

which integrate a vector over a curve. The above equation is really just shorthand notation for three different integrals, each of which expresses a scalar component of the vector $\vec{I}$:

$$I_x = \int_C V_x(x, y, z) \, ds,$$

$$I_y = \int_C V_y(x, y, z) \, ds,$$

and

$$I_z = \int_C V_z(x, y, z) \, ds,$$

which are ordinary line integrals of scalars.

The line integrals treated in this chapter involve either a scalar or vector product of a vector field $\vec{V}(x, y, z)$ with a vector element of integration $d\vec{s}$ defined below. The integrals are written respectively as

$$I = \int_C \vec{V}(x, y, z) \cdot d\vec{s} \quad (7.1)$$
and
\[ \vec{I} = \int_C \vec{V}(x, y, z) \times \vec{ds}. \quad (7.2) \]

Note that the first integral is a scalar and the second is a vector.

The integrals involving the scalar product are used in the definition of work, and, consequently, they are sometimes called “work integrals”, even when they are used for purposes unrelated to work\(^1\). Work integrals are especially important in the theory of electricity because they are used to define the concept of electrical potential, also known as “voltage”. The integrals using the vector product are especially important in the theory of magnetism because they are required to determine the magnetic field produced by an electrical current flowing in a thin wire (Biot-Savart law), and for determining the force exerted by a magnetic field on a current-carrying wire.

For simplicity, we restrict our attention to integrals involving vector fields \( \vec{V}(x, y) \) and curves \( C \) that lie completely within the \( x-y \) plane.

### 7.2 The Vector Element of Integration

The vector element of integration \( \vec{ds} \) for a line integral over a curve \( C \) has magnitude equal to \( ds \) and has direction equal to the direction of the curve at the location of \( ds \) as shown in Figure 7.1. Let the vector \( dx \hat{i} \) be the projection of \( \vec{ds} \) onto the \( x \) axis and the vector \( dy \hat{j} \) be the projection of \( \vec{ds} \) onto the \( y \) axis. Then, as can be seen from Figure 7.2, \( \vec{ds} \) is just the vector sum of these two projections
\[ \vec{ds} = dx \hat{i} + dy \hat{j}. \quad (7.3) \]

\(^1\)This is similar to the use of the term “flux integral” even when none of the quantities involved are flowing.
7.3. WORK INTEGRALS

Hence, $dx$ and $dy$ are respectively the $x$ and $y$ components of $\tilde{ds}$. As such, they can be either positive or negative. For the vector $\tilde{ds}$ illustrated in Figure 7.2, they are both positive. But if the direction of $\tilde{ds}$ were reversed, then both $dx$ and $dy$ would be negative. This is in direct contrast to the situation in all of the previous chapters, where we required that $dx$ and $dy$ always be positive. When we apply Eqn. (7.3) to convert the line integral into definite integrals, we need to pay close attention to the direction of integration along the curve and then integrate along the $x$ and $y$ axes in the corresponding directions. This is discussed in more detail in the appropriate sections.

Finally, it should be noted that $dx$ is zero wherever $C$ is parallel to the $x$ axis and $dy$ is zero wherever $C$ is parallel to the $y$ axis. It is important to keep this in mind when working with line integrals along straight lines that are parallel to the coordinate axes.

7.3 Work Integrals

7.3.1 Introduction

We now consider line integrals of the form

$$I = \int_C \vec{V}(x, y) \cdot d\vec{s}.$$  

Even though work integrals involve a vector in their integrand, they are still line integrals of a scalar, and therefore, belong to the same class of integrals that we discussed in Chapter 4. To display the scalar explicitly, we again apply the definition of the scalar product of two vectors $\vec{A}$ and $\vec{B}$,

$$\vec{A} \cdot \vec{B} = AB \cos \phi,$$

where $\phi$ is the angle between the two vectors when they are placed tail to tail. With this representation of the scalar product, our work integral becomes

$$I = \int_C V(x, y) \cos \phi(x, y) \, ds. \quad (7.4)$$

This form makes it clear that the work integral is just a special case of the line integral of a scalar, which we studied in Chapter 4, except that we again have the new feature we first encountered with flux integrals. That is, the scalar integrand is dependent on the geometry of the curve because of the presence of the factor $\cos \phi(x, y)$. Because of this factor, the scalar integrand changes when we change the curve $C$ even if we do not change the vector field.

In particular, if we reverse the direction of integration along the curve $C$, we get the opposite result from the one we obtained in Chapter 4. In the previous case, the integrand $f(x, y, z)$ was taken to be independent of the curve $C$ so the change in direction made no change in value of the integral. In the case of work integrals, the change in direction causes a change in sign in the
value of the integral. This change is caused by the presence of the \( \cos \phi \) in the integrand which switches sign when the extra 180 degrees are added to the angle \( \phi \). Nothing else is changed by the switch of direction. Keep in mind that this change in sign when we change the direction of integration occurs because of a resulting change in the integrand, not because of a change in the element of integration \( ds \) (as occurs in definite integrals). Since \( ds \) is the magnitude of the vector \( \hat{d}s \), it is always positive.

### 7.3.2 Simple Special Cases

#### Simple Curves

Work integrals simplify greatly when the curve \( C \) is a portion of a straight line that is parallel to one of the coordinate axes. Suppose, for example, that \( C \) passes from \( x = x_i \) to \( x = x_f \) along the straight line \( y = K \), where \( K \) is a constant. Since this line is parallel to the \( x \) axis, \( \hat{d}s \) points in the positive \( x \) direction when \( x_f > x_i \) and in the negative direction when \( x_f < x_i \). This means that we can set

\[
\hat{d}s = \hat{i} \, dx,
\]

since, if we integrate with respect to \( x \) from \( x_i \) to \( x_f \), then \( dx \) is positive when \( x_f > x_i \) and is negative when \( x_f < x_i \). When we do this, our work integral becomes

\[
I = \int_C \vec{V}(x, y) \cdot \hat{d}s = \int_C \vec{V}(x, y) \cdot \hat{i} \, dx = \int_{x_i}^{x_f} V_x(x, K) \, dx.
\]

Note that in the \( x \)-integral, we are integrating in the same direction as \( C \) independent of whether \( x_f \) is larger or smaller than \( x_i \).

#### Simple Integrand

A very important special case, which we will encounter frequently in our studies, is when the magnitude \( V \) of the vector \( \vec{V} \) and the angle \( \phi \) between that vector and \( \hat{d}s \) are both constant along the entire curve. Then, our work integral becomes

\[
I = V \cos \phi \int_C ds = V \cos \phi L, \tag{7.5}
\]

where \( L \) is the length of the curve.

**Sample Problem 7.1** Determine the work done by kinetic friction on a particle of mass \( m \) when it slides across a flat, horizontal surface with coefficient of kinetic friction \( \mu_k \) along a curve \( C \) in the \( x-y \) plane with total length \( L \).

**Solution:**

**Givens:** \( m, \mu_k, L \).
7.3. WORK INTEGRALS

Work due to a force \( \vec{F}(x, y) \) acting on a particle as it moves along a curve \( C \) is given by

\[
W = \int_C \vec{F}(x, y) \cdot \vec{ds} = \int_C F(x, y) \cos \phi(x, y) \, ds.
\]

The magnitude of the kinetic friction force in this problem is \( F(x, y) = f_k = \mu_k N \), where \( N \) is the magnitude of the normal force of the flat surface acting on the particle. Since the flat surface is horizontal, the magnitude of the normal force is just \( mg \), where \( g \) is the acceleration of gravity. The direction of the force is in the opposite direction from the motion of the particle along the curve. Hence, \( \phi(x, y) = \pi \) radians, and \( \cos \phi(x, y) = -1 \). With these values, our work integral becomes

\[
W = -\mu_k mg \int_C ds = -\mu_k mg L.
\]

Notice, that we were able to solve the problem without even knowing the shape of the curve \( C \).

7.3.3 Conversion to a Definite Integral

If we know the magnitude \( V(x, y) \) and the angle \( \phi(x, y) \), then, we can use the theory of Chapter 4 to convert the line integral into a definite integral. In particular, if the curve \( C \) is specified (apart from its direction) by the equation \( y = y(x) \) for \( a_x \leq x \leq b_x \), then we can use Eqn. (4.18) to write

\[
\int_C V(x, y) \cos \phi(x, y) \, ds = \int_{a_x}^{b_x} V[x, y(x)] \cos \phi[x, y(x)] \sqrt{1 + \left( \frac{dy(x)}{dx} \right)^2} \, dx.
\]

Likewise, if the curve \( C \) is specified (apart from its direction) by the equation \( x = x(y) \) for \( a_y \leq y \leq b_y \), then we can use Eqn. (4.19) to write

\[
\int_C V(x, y) \cos \phi(x, y) \, ds = \int_{a_y}^{b_y} V[x(y), y] \cos \phi[x(y), y] \sqrt{1 + \left( \frac{dx(y)}{dy} \right)^2} \, dy.
\]

Remember that the \( x \) and \( y \) integrals in these equations must always be done in the positive direction regardless of the direction of the curve \( C \). If the direction of the curve \( C \) is reversed, the only change in the integrals would be that the angle \( \phi \) includes an extra 180 degrees, which just changes the sign of the result.

This form of the work-integral is not often useful because the angle \( \phi(x, y) \) is usually either unknown or it is a very complicated function. There are situations, however, when it can be the most helpful form, such as the simplest case discussed above in the previous section.
7.3.4 The Differential Form

Another representation of the scalar product of the vectors \( \vec{A} \) and \( \vec{B} \) expresses them in terms of their scalar components

\[
\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z.
\]

Using this representation and the relation for the vector element of integration \( d\vec{s} = dx \hat{i} + dy \hat{j} \), we can write our work integral in the form

\[
I = \int_C [V_x \, dx + V_y \, dy] = \int_C V_x(x, y) \, dx + \int_C V_y(x, y) \, dy. \tag{7.6}
\]

This form of the work integral is called the “differential form”. The first integral on the far right indicates an integral with respect to \( x \) from the initial point \((x_i, y_i)\) at the beginning of the curve to the final point \((x_f, y_f)\) at the end of the curve while \( y \) varies with \( x \) according to the equation for the curve \( y = y(x) \). Likewise, the second integral indicates an integral with respect to \( y \) from the initial point to the final point while \( x \) varies with \( y \) according to the equation for the curve \( x = x(y) \). Since the curve \( C \) is the same for both integrals, the function \( x(y) \) must be the inverse of the function \( y(x) \) obtained by solving the equation \( y = y(x) \) for \( x \) as a function of \( y \). Hence, our work integral can be written as

\[
I = \int_C \vec{V} \cdot d\vec{s} = \int_{x_i}^{x_f} V_x[x, y(x)] \, dx + \int_{y_i}^{y_f} V_y[x(y), y] \, dy. \tag{7.7}
\]

It should be noted that, if \( C \) is a line of constant \( y \), then \( dy = 0 \) and \( d\vec{s} = dx \hat{i} \) along \( C \). Consequently, the second integral on the far right of Eqns. (7.6) and (7.7) should be taken to be zero. Likewise, if \( C \) is a line of constant \( x \), then the first integral on the far right of Eqns. (7.6) and (7.7) should be taken to be zero.

When applying Eqn. (7.7), it is important to keep in mind that the integrals involved differ from all of the other \( x \) and \( y \) integrals that we have considered in this book in that these integrals will sometimes be taken in the negative direction. In particular, according to the way it is written, the \( x \) integral is taken in the negative direction whenever \( x_f \) is smaller than \( x_i \). Likewise, the \( y \) integral is taken in the negative direction whenever \( y_f \) is smaller than \( y_i \). This caution also works the other way as well. After working with these integrals for awhile, it is easy to forget when returning to one of the \( x \) or \( y \) integrals of the previous sections, that they should not be integrated in the negative direction. In particular, if the \( x \) or \( y \) integral in Sec. 7.3.3 is applied to calculate a work integral instead of the integrals in Eqn. (7.7), then you must remember to integrate only in the positive direction, regardless of the direction of the curve \( C \). In that case, the effect of reversing the direction of \( C \) occurs in the \( \cos \phi \) term in the integrand rather than in reversing the direction of the \( x \) or \( y \) integration as happens in Eqn. (7.7).
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If we are given the function \( y(x) \) but are not given the function \( x(y) \), then, in order to apply Eqn. (7.7), we must first invert \( y = y(x) \) to find \( x(y) \). This can be avoided by changing the variable of integration in the \( y \) integral to \( x \) by using \( dy = \frac{dx}{dx} \cdot dx \). When we do this, we must evaluate the integrand \( V_y(x, y) \) on the curve \( C \) by using \( y = y(x) \) instead of \( x = x(y) \). The result is

\[
I = \int_C \vec{V}(x, y) \cdot d\vec{s} = \int_{x_i}^{x_f} \left[ V_x[x, y(x)] + \frac{dy(x)}{dx} V_y[x, y(x)] \right] dx. \tag{7.8}
\]

Likewise, if we are given the function \( x(y) \) but are not given the function \( y(x) \), then, we can change the variable of integration in the \( x \) integral in Eqn. (7.7) to \( y \) by using \( dx = \frac{dx}{dy} \cdot dy \) to obtain

\[
I = \int_C \vec{V}(x, y) \cdot d\vec{s} = \int_{y_i}^{y_f} \left[ \frac{dx(y)}{dy} V_x[x(y), y] + V_y[x(y), y] \right] dy. \tag{7.9}
\]

Sample Problem 7.2 Evaluate the work integral with \( \vec{V} = A \left[ y(\sin kx) \hat{i} + \frac{\cos kx}{k} \right] \) and the curve \( C \) is given by \( y(x) = B \sin kx \) for \( x \) ranging from \( x_i \) to \( x_f \), where \( A, B, k \) are all given constants.

Solution:
Given: \( A, B, k \).

Because we have

\[
V_x[x, y(x)] = Ay \sin kx = AB \sin^2 kx,
\]

\[
V_y[x, y(x)] = A \frac{\cos kx}{k},
\]

and

\[
\frac{dy(x)}{dx} = Bk \cos kx
\]

on the curve \( C \), the integrand of Eqn. (7.8) is

\[
V_x[x, y(x)] + \frac{dy(x)}{dx} V_y[x, y(x)] = AB(\sin^2 x + \cos^2 x) = AB
\]

and the work integral becomes

\[
I = \int_C \vec{V}(x, y) \cdot d\vec{s} = AB \int_{x_i}^{x_f} dx = AB(x_f - x_i).
\]

Sample Problem 7.3 Evaluate the work integral with \( \vec{V} = Ay \hat{i} + Ax \hat{j} \) and the curve \( C \) is a parabolic arc given by \( y(x) = x - Bx^2 \) starting at \( x = x_i \) and ending at \( x = x_f \), where \( A, B, x_i \) and \( x_f \) are all given constants such that \( 0 < x_f < x_i < 1/B \) as shown in Figure 7.3.
CHAPTER 7. LINE INTEGRALS INVOLVING VECTORS

Solution:
Givens: \( A, B, x_i, x_f. \)

Since
\[
V_x[x, y(x)] = Ay(x) = A(x - Bx^2),
\]
\[
V_y[x, y(x)] = Ax,
\]
and
\[
\frac{dy(x)}{dx} = 1 - 2Bx
\]
on the curve \( C \), the integrand in Eqn. (7.8) is
\[
V_x[x, y(x)] + \frac{dy(x)}{dx}V_y[x, y(x)] = A(x - Bx^2) + Ax(1 - 2Bx) = A(2x - 3Bx^2).
\]
Hence, the integral becomes
\[
I = \int_C \vec{V}(x, y) \cdot d\vec{s} = A \int_{x_i}^{x_f} (2x - 3Bx^2) \, dx = A \left[ x_f^2 - x_i^2 - B(x_f^3 - x_i^3) \right].
\]

7.4 Work Integrals of Conservative Vectors

7.4.1 Conservative Vectors

For the special cases of work integrals of gravitational and electrostatic forces, the values of the integrals depend only on the end points of the curve \( C \) and not on the shape of the curve in-between. We say that the work is independent of
the path between the two points. When a vector has this property, we say that it is conservative. Hence, both gravitational and electrostatic forces are conservative.\textsuperscript{2} Magnetic forces are conservative too under certain conditions. Since there are significant ramifications that result from a vector being conservative, we need to have a deeper understanding of what this means. In this section, we consider two-dimensional curves and conservative vector fields that are confined to the $x$-$y$ plane.

A conservative vector field satisfies all three of the statements that follow. These statements are all equivalent to each other. If a vector field satisfies one of them, then it is conservative and satisfies all three.

1. The work integral
   \[
   I = \int_C \vec{V}(x, y) \, dx
   \]
   of the vector $\vec{V}(x, y)$ depends only on the end points of the curve $C$ and is independent of the path taken by $C$ between those points.

2. The components of the vector satisfy
   \[
   \frac{\partial V_x(x, y)}{\partial y} = \frac{\partial V_y(x, y)}{\partial x}. \tag{7.10}
   \]

3. The vector field has a "potential function" $U(x, y)$ which is related to the components of the vector by\textsuperscript{3}
   \[
   V_x(x, y) = -\frac{\partial U(x, y)}{\partial x}, \tag{7.11}
   \]
   \[
   V_y(x, y) = -\frac{\partial U(x, y)}{\partial y}. \tag{7.12}
   \]

Statement 2 provides a very convenient check to determine whether or not a given vector field is conservative. All we need to do is take the partial derivative of the vector’s $x$ component with respect to $y$, take the partial derivative of the vector’s $y$ component with respect to $x$, and compare the two to see whether or not they are equal to each other for all $x$ and $y$.

In the next section, we apply Statements 1 and 3 to obtain the result about work integrals of conservative vectors that is most important to us.

\subsection{Fundamental Theorem of Line Integrals}

Statement 1 above is of great value when we want to evaluate a work integral for a given conservative vector field along a given curve $C$ because it tells us that

\textsuperscript{2}Electric forces lose their conservatism in the presence of time-varying magnetic fields.

\textsuperscript{3}Mathematicians usually do not use the minus signs that appear in these equations when they define the potential function $U$, but physicists do. This is a trivial difference in definition which leads to potential functions that differ only in their sign.
we can integrate it along any other curve we wish as long as it starts at the same initial point and ends at the same final point. So, if we can find another such curve which makes the integration easier to do, we are free to do so, knowing that we will still get the correct result. In particular, it is very convenient to choose a curve that consists of two legs, each of which is a straight line. The first line $C_1$ is parallel to the $x$ axis, going from the initial point $(x_i, y_i)$ to the point $(x_f, y_i)$ directly below or above the final point in the $x$-$y$ plane. The second leg starts at that point and follows a straight line $C_2$ parallel to the $y$ axis until it reaches the final point $(x_f, y_f)$. An example for a specific curve $C$ is shown in Figure 7.4. The new curve makes the integration much easier because, in the expression for the element of integration

$$d\vec{s} = dx \hat{i} + dy \hat{j},$$

dy is zero along the first leg and $dx$ is zero along the second leg. Consequently, when we write the integral $I$ in differential form, we obtain

$$\int_C \vec{V}(x, y) \cdot d\vec{s} = I_1 + I_2,$$

where

$$I_1 = \int_{x_i}^{x_f} V_x(x, y_i) \, dx,$$

and

$$I_2 = \int_{y_i}^{y_f} V_y(x_f, y) \, dy.$$

We now make use of Statement 3 above to express the integrands in the latter two integrals in terms of the potential function of the conservative vector. We
have
\[ V_x(x, y_i) = -\frac{\partial U(x, y_i)}{\partial x}, \]
\[ V_y(x_f, y) = -\frac{\partial U(x_f, y)}{\partial y}. \]

Since \( y_i \) is a fixed constant in the first equation and \( x_f \) is a fixed constant in the second one, the above partial derivatives are the same as ordinary 1-dimensional derivatives. When we substitute these equations into the expressions for \( I_1 \) and \( I_2 \), we find
\[ I_1 = \int_{x_i}^{x_f} -\frac{dU(x, y_i)}{dx} \, dx = -[U(x_f, y_i) - U(x_i, y_i)], \]
and
\[ I_2 = \int_{y_i}^{y_f} -\frac{dU(x_f, y)}{dy} \, dy = -[U(x_f, y_f) - U(x_f, y_i)]. \]

When we add these two expressions together, we obtain our final expression
\[
\int_C \vec{V}(x, y) \cdot d\vec{s} = -[U(x_f, y_f) - U(x_i, y_i)]. \tag{7.13}
\]

This is a very important result known as the Fundamental Theorem of Line Integrals. It tells us how to evaluate a work integral of a conservative vector in terms of its potential function at the two end points of the curve \( C \) without requiring us to perform any integration. All we need to do is to evaluate the potential function for the vector \( \vec{V} \) at the two end points of the curve \( C \) and subtract the results. Note that it is the two-dimensional analog of the Fundamental Theorem of Calculus which states that, if the integrand \( f(x) \) of a one-dimensional definite integral with respect to \( x \) can be expressed as the derivative of a function \( g(x) \) with respect to \( x \), then
\[
\int_{x_i}^{x_f} f(x) \, dx = g(x_f) - g(x_i).
\]

Just as the above formula is extremely valuable for evaluating definite integrals, the Fundamental Theorem of Line Integrals is extremely valuable for evaluating work integrals of conservative vectors. As you know, the main difficulty with attempting to apply the Fundamental Theorem of Calculus is that we need to find the function \( g(x) \) that satisfies \( \frac{dg(x)}{dx} = f(x) \). Likewise, in order to apply Fundamental Theorem of Line Integrals, we need to find the potential function \( U(x, y) \) that gives minus the \( x \) component of the vector field when it is differentiated partially with respect to \( x \) and gives minus the \( y \) component of the vector field when it is differentiated partially with respect to \( y \) according to Statement 3. Before attempting to do that, it is usually wise to differentiate the \( x \) component of the vector with respect to \( y \) and the \( y \) component of the vector with respect to \( x \) to make sure that the two partial derivatives are equal.
as required by Statement 2. If the derivatives are not equal, then there is no point in trying to find the potential function because it doesn’t exist (since the vector is not conservative). In that case, it is necessary to use the methods of Section 7.3 to evaluate the integral.

There is a straightforward way to determine the potential function of a given conservative vector field. First, we take the indefinite integral of both sides of Eqn. (7.11)

\[ V_x(x, y) = -\frac{\partial U(x, y)}{\partial x} \]

with respect to \( x \) holding \( y \) constant. The result is

\[ U(x, y) = -\int V_x(x, y) \, dx + f(y). \quad (7.14) \]

The quantity \( f(y) \) is an unknown “constant” of integration. But since \( y \) is held constant in the above indefinite integral, the “constant” of integration is constant only with respect to \( x \). It can vary with respect to \( y \). In order to figure out how it varies with \( y \), we take the indefinite integral of both sides of Eqn. (7.12)

\[ V_y(x, y) = -\frac{\partial U(x, y)}{\partial y}. \]

with respect to \( y \) holding \( x \) constant. The result is

\[ U(x, y) = -\int V_y(x, y) \, dy + g(x). \quad (7.15) \]

This time, the “constant” of integration can depend on \( x \). Since the right-hand sides of the above two equations must be equal, and since \( f(y) \) has no \( x \) dependence and \( g(x) \) has no \( y \) dependence, it is easy to see what \( f(y) \) is or what \( g(x) \) is apart from an arbitrary constant \( K \) by comparing the two right-hand sides. It is easiest to see how to do this by looking at explicit formulas, for example, in a Sample Problem.

**Sample Problem 7.4** Evaluate the work integral with \( \vec{V} = Ay\hat{i} + Ax\hat{j} \) and the curve \( C \) is a parabolic arc given by \( y(x) = x - Bx^2 \) starting at \( x = x_i \) and ending at \( x = x_f \), where \( A, B, x_i \) and \( x_f \) are all given constants such that \( 0 < x_f < x_i < 1/B \) as shown in Figure 7.3.

**Solution:**

**Givens:** \( A, B, x_i, x_f \).

This is the same problem as we did in Sample Problem 7.3 except this time, we will use the methods of this section. Since these methods are useful only if the vector \( \vec{V} \) is conservative, we should check to see if it is. To that end, we differentiate the \( x \) component of the vector with respect to \( y \) and the \( y \) component of the vector with respect to \( x \). According to Statement 2, the vector is conservative if these two derivatives are equal. Upon doing the derivatives, we
see that both of them are equal to $A$ and therefore conclude that the vector is conservative.

Next, we attempt to determine the potential function for our vector $\vec{V}$ using the approach described above. We first integrate the $x$ component of the vector with respect to $x$ holding $y$ constant according to Eqn. (7.14). Since the $x$ component of the vector is given to be $Ay$, the result is

$$U(x, y) = -\int Ay \, dx + f(y) = -Axy + f(y).$$

where $f(y)$ is an undetermined function of $y$ that does not depend on $x$. Next, we integrate the $y$ component of the vector with respect to $y$ holding $x$ constant according to Eqn. (7.15). Since the $y$ component of the vector is given to be $Ax$, the result is

$$U(x, y) = -\int Ax \, dy + g(x) = -Axy + g(x).$$

where $g(x)$ does not depend on $y$. From the second of these two equations, we see that the only $y$ dependence that $U(x, y)$ has is $Axy$. Therefore, $f(y)$ in the first equation can not depend on $y$. It must be a true constant $K$. This means that the first equation can be written

$$U(x, y) = -Axy + K,$$

where $K$ is an arbitrary constant.

Now that we know the potential function, we are ready to apply the Fundamental Theorem of Line Integrals

$$\int_C \vec{V}(x,y) \cdot \, d\vec{s} = -(U(x_f, y_f) - U(x_i, y_i)).$$

With the application of potential function we found above, this equation gives

$$\int_C \vec{V}(x,y) \cdot \, d\vec{s} = A[x_f y_f - x_i y_i].$$

This is not a symbolic solution to our problem yet, because $y_i$ and $y_f$ are not givens. We need to express them in terms of the givens $x_i$ and $x_f$. Since the curve $C$ is the parabolic arc $y = y(x) = x - Bx^2$, we have $y_i = x_i - Bx_i^2$ and $y_f = x_f - Bx_f^2$. Substituting these results into the above equation gives

$$\int_C \vec{V}(x,y) \cdot \, d\vec{s} = A[x_f(x_f - Bx_f^2) - x_i(x_i - Bx_i^2)].$$

This is the same result as we obtained in Sample Problem 7.3. Note that for this problem, the current method was a little more difficult to do than evaluating the integral directly as done in the previous Sample Problem. But if the curve $C$ is changed to a more complicated curve with the same end points, direct integration would probably become more difficult (perhaps even impossible), whereas the current method would not be affected at all.
7.5 Line Integrals Involving the Vector Product

7.5.1 Introduction
We now consider line integrals of the form
\[ \vec{I} = \int_C \vec{V}(x, y, z) \times ds. \]
This time the integrand is a vector. However, since we continue to restrict our attention to vectors \( \vec{V} \) and curves \( C \) that both lie in the \( x-y \) plane, both vectors in the integrand lie in that plane. This means that the vector product of those vectors must always be perpendicular to the \( x-y \) plane. Since the magnitude of the vector product of vectors \( \vec{A} \) and \( \vec{B} \) is \( AB \sin \phi \), where \( \phi \) is the angle in between them when they are tail to tail, our integral can be written
\[ \vec{I} = \hat{k} \int_C V(x, y) \sin \phi(x, y) \, ds, \]
(7.16)
where \( V(x, y) \) and \( ds \) are both positive. Hence, the sign of the integrand is determined solely by the sign of \( \sin \phi(x, y) \).

The integral on the right-hand side of Eqn. (7.16) is again a special case of the line integral of a scalar, which we studied in Chapter 4, except that we again have the new feature we first encountered with flux integrals. That is, the scalar integrand is dependent on the geometry of the curve because of the presence of the factor \( \sin \phi(x, y) \). Because of this factor, the scalar integrand changes when we change the curve \( C \) even if we do not change the vector field. Because \( \sin \phi \) simply changes sign when 180 degrees are added to \( \phi \), we see that this integral behaves similarly to work integrals if we reverse the direction of the curve \( C \). The only change is that the sign of the integral is changed.

7.5.2 The Simplest Case
Again, we consider the important case when the magnitude \( V \) of the vector \( \vec{V} \) and the angle \( \phi \) are constant along the entire curve. Then, our integral becomes
\[ \vec{I} = \hat{k}V \sin \phi \int_C ds = \hat{k}V \sin \phi L, \]
(7.17)
where \( L \) is the length of the curve.

7.5.3 Conversion to a Definite Integral
If we know the magnitude \( V(x, y) \) and the angle \( \phi(x, y) \), then, we can use the theory of Chapter 4 to convert the line integral into a definite integral. In particular, if the curve \( C \) is specified (apart from its direction) by the equation \( y = y(x) \) for \( a_x \leq x \leq b_x \), then we can use Eqn. (4.18) to write
\[ \vec{I} = \hat{k} \int_C V(x, y) \sin \phi(x, y) \, ds = \hat{k} \int_{a_x}^{b_x} V[x, y(x)] \sin \phi[x, y(x)] \sqrt{1 + \left( \frac{dy(x)}{dx} \right)^2} \, dx. \]
Likewise, if the curve $C$ is specified (apart from its direction) by the equation $x = x(y)$ for $a_y \leq x \leq b_y$, then we can use Eqn. (4.19) to write

$$\vec{I} = \hat{k} \int_C V(x, y) \sin \phi(x, y) \, ds = \hat{k} \int_{a_y}^{b_y} V[x(y), y] \sin \phi[x(y), y] \sqrt{1 + \left(\frac{dx(y)}{dy}\right)^2} \, dy.$$ 

Remember that the $x$ and $y$ integrals in these equations must always be done in the positive direction regardless of the direction of the curve $C$. If the direction of the curve $C$ is reversed, the only change in the integrals would be that the angle $\phi$ includes an extra 180 degrees, which just changes the sign of the result.

This form of the integral is not often useful because the angle $\phi(x, y)$ is usually either unknown or it is a very complicated function. There are situations, however, when it can be the most helpful form, such as the simplest case discussed above in the previous section.

### 7.5.4 The Differential Form

If the vectors $\vec{A}$ and $\vec{B}$ both lie in the $x$-$y$ plane, their cross product can be expressed in terms of their components as

$$\vec{A} \times \vec{B} = (A_x B_y - A_y B_x) \hat{k}.$$ 

Hence, since the vector $\vec{ds}$ has $x$ component equal to $dx$ and $y$ component equal to $dy$, we can write

$$\vec{I} = \int_C \vec{V} \times \vec{ds} = \hat{k} \int_C \left[ V_x(x, y) \, dy - V_y(x, y) \, dx \right] = \hat{k} \left[ \int_C V_x(x, y) \, dy - \int_C V_y(x, y) \, dx \right].$$  

(7.18)

Following the terminology adopted for work integrals, we call this the differential form of our cross product integral.

The first integral on the far right indicates an integral with respect to $y$ from the initial point $(x_i, y_i)$ at the beginning of the curve to the final point $(x_f, y_f)$ at the end of the curve while $x$ varies with $y$ according to the equation for the curve $x = x(y)$. Likewise, the second integral indicates an integral with respect to $x$ from the initial point to the final point while $y$ varies with $x$ according to the equation for the curve $y = y(x)$. Since the curve $C$ is the same for both integrals, the function $x(y)$ must be the inverse of the function $y(x)$ obtained by solving the equation $y = y(x)$ for $x$ as a function of $y$. Hence, our integral can be written as

$$\vec{I} = \int_C \vec{V} \times \vec{ds} = \hat{k} \left( \int_{y_i}^{y_f} V_x[x(y), y] \, dy - \int_{x_i}^{x_f} V_y[x, y(x)] \, dx \right).$$  

(7.19)

It should be noted that, if $C$ is a line of constant $y$, then $dy = 0$ and $\vec{ds} = dx \hat{i}$ along $C$. Consequently, the second integral on the far right of Eqns. (7.18) and (7.19) should be taken to be zero. Likewise, if $C$ is a line of constant $x$, then
CHAPTER 7. LINE INTEGRALS INVOLVING VECTORS

the first integral on the far right of Eqns. (7.18) and (7.19) should be taken to be zero.

When applying Eqn. (7.19) it is important to keep in mind that these integrals will sometimes be taken in the negative direction. In particular, according to the way it is written, the $x$ integral is taken in the negative direction whenever $x_f$ is smaller than $x_i$. Likewise, the $y$ integral is taken in the negative direction whenever $y_f$ is smaller than $y_i$.

If we are given the function $y(x)$ but are not given the function $x(y)$, then, in order to apply Eqn. (7.18), we must first invert $y = y(x)$ to find $x(y)$. This can be avoided by changing the variable of integration in the $y$ integral to $x$ by using $dy = \frac{dy(x)}{dx} dx$. When we do this, we must evaluate the integrand $V_x(x, y)$ on the curve $C$ by using $y = y(x)$ instead of $x = x(y)$. The result is

$$\vec{I} = \int_C \vec{V}(x, y) \times d\vec{s} = \hat{k} \int_{x_i}^{x_f} \left( V_x[x, y(x)] \frac{dy(x)}{dx} - V_y[x, y(x)] \right) dx. \quad (7.20)$$

Likewise, if we are given the function $x(y)$ but are not given the function $y(x)$, then, we can change the variable of integration in the $x$ integral in Eqn. (7.18) to $y$ by using $dx = \frac{dx(y)}{dy} dy$ to obtain

$$\vec{I} = \int_C \vec{V}(x, y) \times d\vec{s} = \hat{k} \int_{y_i}^{y_f} \left( V_x[x(y), y] - V_y[x(y), y] \frac{dx(y)}{dy} \right) dy. \quad (7.21)$$

Sample Problem 7.5 Evaluate the integral

$$\vec{I} = \int_C \vec{V}(x, y) \times d\vec{s}$$

with $\vec{V} = Ay\hat{i} + Ax\hat{j}$ and the curve $C$ is a parabolic arc given by $y(x) = x - Bx^2$ starting at $x = x_i$ and ending at $x = x_f$, where $A, B, x_i$ and $x_f$ are all given constants such that $0 < x_f < x_i < 1/B$ as shown in Figure 7.3.

Solution:

Given: $A, B, x_i, x_f$.

This problem has the same vector $\vec{V}$ and the same curve $C$ as those in Sample Problem 7.5, but this time, the integral to be evaluated involves a vector product instead of a scalar product. Because we are given $y(x)$ to describe the curve $C$, we choose to work with Eqn. (7.20)

$$\vec{I} = \int_C \vec{V}(x, y) \times d\vec{s} = \hat{k} \int_{x_i}^{x_f} \left( V_x[x, y(x)] \frac{dy(x)}{dx} - V_y[x, y(x)] \right) dx.$$

Since

$$V_x[x, y(x)] = Ay(x) = A(x - Bx^2),$$

$$V_y[x, y(x)] = Ax,$$
and 
\[ \frac{dy(x)}{dx} = 1 - 2Bx \]
on the curve \( C \), the integrand is 
\[ V_x[x, y(x)] \frac{dy(x)}{dx} - V_y[x, y(x)] = A(x - Bx^2)(1 - 2Bx) - Ax = A(2B^2x^3 - 3Bx^2). \]
Hence, the integral becomes 
\[ \vec{I} = A\hat{k} \int_{x_i}^{x_f} (2B^2x^3 - 3Bx^2) \, dx = A\hat{k} \left[ \frac{1}{2} B^2 (x_i^4 - x_f^4) - B(x_i^3 - x_f^3) \right]. \]
CHAPTER 7. LINE INTEGRALS INVOLVING VECTORS

PROBLEMS

1. A particle moves along the path \( y = Ax^2 \) from \( x = x_i \) to \( x = x_f \), where \( x_i \leq x \leq x_f \). As it moves, it experiences a force \( \vec{F}(x, y) \), which has magnitude \( F(x, y) = B\sqrt{x\sqrt{y}} \). The direction of the force is always directly opposite from the direction of the particle’s motion. Obtain a symbolic solution for the work done by the force in terms of the given constants \( A, B, x_i, \) and \( x_f \). (Solution check: The numerical result with \( A = 16.0 \text{ m}^{-1}, B = 144 \text{ N m}^{-3/4}, x_i = 0.00 \text{ m}, \) and \( x_f = 2.00 \text{ m} \) is \(-2.46 \times 10^4 \text{ J}\).)

2. Evaluate the work integral of the vector \( \vec{V} = Bx^2\hat{i} + x^3\hat{j} \) over the curve \( C \) given by \( y = Ax^3 \), starting at \( x_i \) and ending at \( x_f \), where \( A \) is a positive constant. (Solution check: The numerical value with \( B = 100 \text{ m}^2, A = 3.00 \text{ m}^{-2}, x_i = 1.00 \text{ m}, \) and \( x_f = 2.00 \text{ m} \) is \(3.08 \times 10^4 \text{ m}^6\).)

3. Find the potential function \( U(x, y) \) for the conservative vector \( \vec{V} = -ke^{xy}y^2\hat{i} + (\hat{-}2e^{xy} + Ak \cos ky)\hat{j} \), where \( k \) and \( A \) are constants. (Solution check: You can easily check your answer without knowing any numerical values.)

4. Evaluate the work integral of the vector \( \vec{V}(x, y) = 2(x + y)\hat{i} + 2(x + y)\hat{j} \) over a smooth curve from \( (x_i, y_i) \) to \( (x_f, y_f) \) in terms of the given constants \( x_i, y_i, x_f, \) and \( y_f \). (Solution check: The numerical value with \( x_i = 3.00 \text{ m}, y_i = 5.00 \text{ m}, x_f = 2.00 \text{ m}, \) and \( y_f = 4.00 \text{ m} \) is \(-28.0 \text{ m}^2\).)

5. Evaluate the integral \( I = \int_C \vec{V} \times d\vec{s} \), where the vector \( \vec{V} \) and the curve \( C \) are the same as in Problem 2. (Solution check: The numerical result with \( B = 100 \text{ m}^2, A = 3.00 \text{ m}^{-2}, x_i = 1.00 \text{ m}, \) and \( x_f = 2.00 \text{ m} \) is \(8.51 \times 10^4k \text{ m}^6\).)

6. (Supplemental problem for students who have studied the chapter in their physics text that deals with magnetic forces on current-carrying wires.) A very thin wire, which follows a semicircular curve \( C \) of radius \( R \), lies in the upper half of the \( x-y \) plane with its center at the origin. There is a constant current \( I \) flowing counter clockwise, starting upward from the end of the wire on the positive \( x \) axis and ending downward at the end on the negative \( x \) axis. The wire is in a uniform magnetic field, which has magnitude \( B_0 \) and direction parallel to the \( z \) axis in the positive \( z \) direction. Determine a symbolic answer in unit-vector notation for the total force on the wire due to the magnetic field. Ignore the forces on the leads that carry the current into the wire at the right end and out of the wire at the left end. HINT: Since the vector \( \vec{B} \) does not lie in the \( x-y \) plane, you will need to modify the analysis given in Sec. 7.5.4. (Solution check: The numerical value with \( I = 2.00 \text{ A}, B_0 = 3.00 \text{ T}, \) and \( R = 4.00 \text{ m} \) is \(48.0\hat{j} \text{ N}\).)

7. (Supplemental problem for students who have studied the chapter in their physics text that deals with Ampere’s law.) Consider a magnetic field
7.5. **LINE INTEGRALS INVOLVING THE VECTOR PRODUCT**

which varies with position according to \( \vec{B}(x, y) = K(x^4 y^3 \hat{i} - x^3 y^4 \hat{j}) \), where \( K \) is a positive constant. Find the net current flowing in the positive z direction (out of the paper) through the area outlined by the triangle in Figure 7.5. (Solution check: The numerical value with \( K = 3 \ Tm^{-7} \), \( x_0 = 2 \ m \), and \( y_0 = 4 \ m \) is \( \frac{-8448}{\mu_0} \ A \).)

![Figure 7.5: Geometry for Problem 7](image1)

![Figure 7.6: Geometry for Problem 8](image2)

8. (Supplemental problem for students who have studied the chapter in their physics text that deals with induction.) Consider an electric field which does not vary with time or the z direction, but which varies in the \( x \) and \( y \) directions according to \( \vec{E} = K(x^2 y \hat{i} - xy^2 \hat{j}) \) throughout the region of interest, where \( K \) is a constant. Show that the magnetic flux \( \Phi_B \) through the triangle shown in Figure 7.6 is given by \( \Phi_B = K \frac{3 + \tan^2 \theta}{12} x_0^4 \tan \theta \ t + c \), where \( c \) is an arbitrary constant and \( t \) represents time.
Appendix A

Triple Integrals

A.1 Introduction

This appendix is concerned with the integration of a function over a three-dimensional volume $V$. The integrals of interest are often referred to as volume integrals and can be written in the form

$$ I = \int_V f(x, y, z) \, dV. \quad (A.1) $$

A notation that acknowledges the three-dimensionality of the integration expresses the volume integral as

$$ I = \iiint_V f(x, y, z) \, dV. \quad (A.2) $$

When written this way, the volume integral is referred to as a triple integral.

Volume integrals are needed occasionally in electromagnetic theory in order to be able to evaluate charge integrals over three-dimensional regions. For example, it is necessary to evaluate a volume integral over a volume $V$ in order to use Coulomb’s law to determine the electric field produced by electric charge that is distributed over that volume in a known way. Also, when applying Gauss’ law to determine the electric flux through a closed surface, it can be necessary to integrate the volume charge density over the volume inclosed by that surface in order to determine the total charge within. These occasions are relatively rare, however, compared to how frequently line integrals and surface integrals are required. In contrast to volume integrals, line and surface integrals occur in Maxwell’s equations and are much more deeply embedded throughout electromagnetic theory. For this reason, the discussion of volume integrals in this book is brief and is relegated to an appendix. Only the simplest of geometries are treated.

In what follows, it is assumed that the reader has already mastered the material on double integrals presented in Chapter 5.
A.2 Definition

The volume integral is defined by the same procedure as the one used to define the area integral in Section 5.2 except that we replace the tiny, flat areas $\Delta A_i$ that make up the flat surface $A$ with tiny volumes $\Delta V_i$ that make up the volume $V$. The defining equation becomes

$$\int_V f(x, y, z) \, dV = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta V_i.$$  \hfill (A.3)

Because the definitions of triple integrals and area integrals are so similar, they share many common properties. For example, we have

$$\int_V Kg(x, y, z) \, dV = K \int_V g(x, y, z) \, dV,$$  \hfill (A.4)

$$\int_V [f(x, y, z) + g(x, y, z)] \, dV = \int_V f(x, y, z) \, dV + \int_V g(x, y, z) \, dV,$$  \hfill (A.5)

$$\int_{V_1+V_2} f(x, y, z) \, dV = \int_{V_1} f(x, y, z) \, dV + \int_{V_2} f(x, y, z) \, dV,$$  \hfill (A.6)

and

$$\int_V K \, dV = KV,$$  \hfill (A.7)

where $K$ is a constant and $V$ is the volume of $V$. These properties follow directly from the definition of volume integrals given in Eqn. (A.3).

As usual, it is not convenient to evaluate volume integrals by using the defining equation Eqn. (A.3). Instead, we need to learn how to express the volume integral in terms of definite integrals. In the sections that follow, we learn how to do this three different ways using three different coordinate systems.

A.3 Evaluation Using Cartesian Coordinates

When using the cartesian coordinates $(x, y, z)$, we choose the little volumes $\Delta V$ to be tiny rectangular boxes with sides parallel to the coordinate planes and with length, width, and heights respectively $\Delta x_i$, $\Delta y_i$, and $\Delta z_i$, all of which are taken to be positive. Then, we have

$$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i,$$  \hfill (A.8)

and the triple integral can be written as

$$I = \int \int \int_V f(x, y, z) \, dx \, dy \, dz.$$  \hfill (A.9)

Hence, our volume integral can be expressed as a triple iterated integral once we specify the shape of $V$ in a way that allows us to specify the limits of integration.
for the $x$, $y$, and $z$ integrals. Since some regions with complicated shapes can prove to be difficult to treat, we restrict our attention to a few simple shapes.

### A.3.1 Rectangular Box

The simplest shape of all is the rectangular box described by

$$\begin{align*}
  a_x &\leq x \leq b_x, \\
  a_y &\leq y \leq b_y, \\
  a_z &\leq z \leq b_z,
\end{align*}$$

where all of the $a$’s and $b$’s are constants. For this shape, the limits of integration are obvious, and our volume integral becomes

$$I = \int_{a_z}^{b_z} \int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x, y, z) \, dx \, dy \, dz. \quad (A.10)$$

This is a triple iterated integral, in which we first evaluate the definite $x$ integral, followed by evaluation of the definite $y$ integral, and completed by evaluation of the definite $z$ integral. Since none of the limits of integration depend on the variables of integration, we can interchange the orders of integration any way we wish without complication. The six possibilities for $dV$ are $(dx \, dy \, dz)$, $(dy \, dz \, dx)$, $(dz \, dx \, dy)$, $(dx \, dz \, dy)$, $(dy \, dx \, dz)$, and $(dz \, dy \, dx)$. When interchanging the order of integration, be sure to switch the limits of integration as well so that the limits of the $x$ integral are always $a_x$ and $b_x$ and similarly for the limits of the $y$ and $z$ integrals.

### A.3.2 $z$-Simple Volumes

We next consider a class of shapes which is more general than the rectangular boxes just discussed. A volume $V$ is said to be $z$-simple if every vertical straight line that intersects the boundary of $V$ does so at most twice, i.e. the line either just grazes the volume without penetration or it passes into the volume and the passes out again without any further penetration. To be more precise, let the area $A$ be the projection of the volume onto the $x$-$y$ plane. Then $V$ is $z$-simple if the upper boundary of the solid is the surface $z = z_+(x, y)$ and the lower boundary is the surface $z = z_-(x, y)$, where $z_-(x, y)$ and $z_+(x, y)$ are continuous functions satisfying $z_-(x, y) \leq z_+(x, y)$ for all $x$ and $y$ in $A$. This means that the upper and lower surfaces can touch each other, but they can not cross each other. For such volumes, the triple iterated integral in Eqn. (A.9) can be written as

$$I = \int_{A} \int_{z_-(x,y)}^{z_+(x,y)} f(x, y, z) \, dz \, dA. \quad (A.11)$$

\(^1\text{When specifying the limits of integration, it is important to keep the element of volume } dV \text{ positive. One way to do this is to choose each of the three integrations to be in the positive direction.}\)
In this case, we do the $z$ integral first, integrating from the bottom surface to the top surface for arbitrary fixed $x$ and $y$ in $A$. Then, we do the remaining area integral over the region $A$ using the methods presented in Section 5.4. If $A$ is $y$-simple, then we do the $y$ integral first, followed by the $x$ integral. If $A$ is $x$-simple, then we do the iterated integral in the opposite order. If $A$ is regular, then we can do the iterated integral in either order. If the area $A$ is either $r$-simple or $\theta$-simple, then we can switch to polar coordinates, if we wish, and apply the methods given in Section 5.5.

If the volume is not $z$-simple but is both $x$-simple and $y$-simple, then a similar formulation with appropriate modifications can be applied to evaluate the volume integral.

**Sample Problem A.1** Determine the volume of a sphere of radius $R$.

**Solution:**

**Givens:** $R$.

From Eqn. (A.7), we have

$$V = \int \int \int_V dV,$$

where $V$ is the sphere of radius $R$ and $V$ is its volume. It is easy to see that the sphere is $z$-simple. Placing the sphere with its center at the origin, the equation for the upper surface is $z = z_+(x, y) = \sqrt{R^2 - x^2 - y^2}$ and the equation for the lower surface is $z = z_-(x, y) = -\sqrt{R^2 - x^2 - y^2}$. Hence, with application of Eqn.(A.11), we obtain

$$V = \int \int_A \left[\int_{z_+}^{z_-} \sqrt{R^2 - x^2 - y^2} \, dz\right] \, dA = 2 \int \int_A \left[\sqrt{R^2 - x^2 - y^2}\right] \, dA,$$

where $A$ is the circular disc $\sqrt{x^2 + y^2} \leq R$. Since the circular disc is $\theta$-simple, we can switch to polar coordinates using $x = r \cos \theta$, $y = r \sin \theta$, and $dA = r \, d\theta \, dr$ to obtain

$$V = 2 \int_0^R \int_0^{2\pi} \sqrt{R^2 - r^2} \, r \, d\theta \, dr = 4\pi \int_0^R \sqrt{R^2 - r^2} \, r \, dr.$$

Finally, we make the $u$-substitution $u = R^2 - r^2$ with $du = -2r \, dr$ to obtain

$$V = -2\pi \int_{R^2}^0 u^{1/2} \, du = \frac{4}{3} \pi R^3.$$

**A.4 Evaluation Using Cylindrical Coordinates**

Consider a point with cartesian coordinates $(x, y, z)$. Cylindrical coordinates express the point’s location by using polar coordinates $r$ and $\theta$ to describe the location of its projection on the $x$-$y$ plane and using $z$ to describe its height.
A.4. EVALUATION USING CYLINDRICAL COORDINATES

above the $x$-$y$ plane as shown in Figure A.1. The two sets of coordinates are related by

\begin{align}
    x &= r \cos \theta, \\
    y &= r \sin \theta, \\
    z &= z.
\end{align}

When a function $f$ of position is expressed in terms of cylindrical coordinates, it is written as $f(r, \theta, z)$.

The tiny volumes $\Delta V$ that are used to define volume integrals in cylindrical coordinates are circular wedges as illustrated in Figure A.2. As $\Delta r$ and $\Delta \theta$ approach zero, the tiny volumes have areas $\Delta A$ that approach $r \Delta r \Delta \theta$ in a plane with constant $z$, and they have heights $\Delta z$. Hence, the element of volume in cylindrical coordinates is

\[ dV = r \, dr \, d\theta \, dz. \]

The simplest shape for integration in cylindrical coordinates is the cylindrical box illustrated in Figure A.3 and described mathematically by

\begin{align}
    0 &\leq a_r \leq r \leq b_r, \\
    0 &\leq a_\theta \leq \theta \leq b_\theta \leq 2\pi,
\end{align}

When specifying the limits of integration, it is important to keep the element of volume $dV$ positive. One way to do this is to choose each of the three integrations to be in the positive direction.
where all of the $a$’s and $b$’s are constants which satisfy the limits given above. For this shape, the limits of integration are obvious, and our volume integral becomes

$$I = \int_{a_z}^{b_z} \int_{a_{\theta}}^{b_{\theta}} \int_{a_r}^{b_r} f(r, \theta, z) \ r \ dr \ d\theta \ dz.$$  \hspace{3cm} (A.16)

This is a triple iterated integral, in which we first evaluate the definite $r$ integral, followed by evaluation of the definite $\theta$ integral, and completed by evaluation of the definite $z$ integral. Since none of the limits of integration depend on the variables of integration, we can interchange the orders of integration any way we wish without complication. The six possibilities for $dV$ are ($r \ dr \ d\theta \ dz$), ($r \ d\theta \ dz \ dr$), ($r \ dz \ dr \ d\theta$), ($r \ dr \ dz \ d\theta$), ($r \ d\theta \ dr \ dz$), and ($r \ dz \ d\theta \ dr$). When interchanging the order of integration, be sure to switch the limits of integration as well so that the limits of the $r$ integral are always $a_r$ and $b_r$ and similarly for the limits of the $\theta$ and $z$ integrals.

When applying cylindrical coordinates, $z$-simple shapes are defined in the same way as when using cartesian coordinates. Moreover, we can still use Eqn. (A.11), except that we rewrite it in terms of cylindrical coordinates;

$$I = \int \int_A \left[ \int_{z_-(r,\theta)}^{z_+(r,\theta)} f(r, \theta, z) \ dz \right] \ dA,$$  \hspace{3cm} (A.17)

where $dA = r \ dr \ d\theta$ or $dA = r \ d\theta \ dr$. In particular, if $A$ is $r$-simple and defined by

$$0 \leq \theta_1 \leq \theta \leq \theta_2 \leq 2\pi$$

and

$$0 \leq r_1(\theta) \leq r \leq r_2(\theta),$$
A.5. EVALUATION USING SPHERICAL COORDINATES

Consider a point with cartesian coordinates \((x, y, z)\). Spherical coordinates express the point’s location by using the distance \(r\) from the origin to the point and the angles \(\theta\) and \(\phi\) as shown in Figure A.4. Note that the notation \(r\) and \(\theta\) represent different quantities than they do in cylindrical coordinates.\(^5\) Careful

\(^{4}\)To make sure that you understand how to apply this equation to simple geometries, use it to solve Sample Problem A.1.

\(^{5}\)Note, also, that the physics convention used here to label the two angles \(\theta\) and \(\phi\) are reversed compared to the convention used by many mathematicians.

then, Eqn(A.17) becomes

\[
I = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \left[ \int_{z_-(r,\theta)}^{z_+(r,\theta)} f(r, \theta, z) \, dz \right] r \, dr \, d\theta. \tag{A.18}
\]

And, if \(A\) is \(\theta\)-simple and defined by

\[
0 \leq r_1 \leq r \leq r_2
\]

and

\[
0 \leq \theta_1(r) \leq \theta \leq \theta_2(r) \leq 2\pi,
\]

then, Eqn(A.17) becomes\(^4\)

\[
I = \int_{r_1}^{r_2} \int_{\theta_1(r)}^{\theta_2(r)} \left[ \int_{z_-(r,\theta)}^{z_+(r,\theta)} f(r, \theta, z) \, dz \right] r \, d\theta \, dr. \tag{A.19}
\]

A.5 Evaluation Using Spherical Coordinates

Figure A.4: Spherical coordinates.
study of Figure A.4 reveals that cartesian and spherical coordinates are related by

\[ x = r \sin \theta \cos \phi, \quad (A.20) \]
\[ y = r \sin \theta \sin \phi, \quad (A.21) \]
\[ z = r \cos \theta. \quad (A.22) \]

When a function \( f \) of position is expressed in terms of spherical coordinates, it is written as \( f(r, \theta, \phi) \).

In order to be able to apply spherical coordinates to evaluate volume integrals, it is important to recognize the surfaces described by holding one of the coordinates constant while varying the other two coordinates. For example, while viewing Figure A.4, imagine holding \( r \) at a constant value \( r = r_0 \) and varying \( \theta \) from 0 to \( \pi \) and varying \( \phi \) from 0 to \( 2\pi \). The surface that is swept out by this procedure is the surface of a sphere of radius \( r_0 \). If, instead, we imagine holding \( \theta \) constant and varying \( r \) from 0 to \( r_0 \) and varying \( \phi \) from 0 to \( 2\pi \), then we sweep out the surface of a circular cone. Finally, if we imagine holding \( \phi \) constant and varying \( r \) from 0 to \( r_0 \) and \( \theta \) from 0 to \( \pi \), we sweep out half of a vertical circular disk of radius \( r_0 \).

Spherical coordinates are most useful for evaluation of volume integrals when the volume is bounded by the surfaces just described. In this appendix, the discussion is limited to volumes that can described mathematically by

\[ 0 \leq a_r \leq r \leq b_r, \quad (A.23) \]
\[ 0 \leq a_\theta \leq \theta \leq b_\theta \leq \pi, \quad (A.24) \]
\[ 0 \leq a_\phi \leq \phi \leq b_\phi \leq 2\pi, \quad (A.25) \]

where all of the \( a \)'s and \( b \)'s are constants which satisfy the limits shown above. This shape can be referred to loosely as a “spherical box”. An example is illustrated in Figure A.5. Note that the front side is a portion of a vertical circular disc with \( \phi \) fixed at \( a_\phi \) and the back side has the same shape, but \( \phi \) is fixed at \( b_\phi \). Similarly, the top side is a section of a circular cone with \( \theta = a_\theta \), and the bottom side is a section of a circular cone with \( \theta = b_\theta \). Finally, the left end is a portion of a spherical surface with \( r = a_r \), and the right end is the corresponding portion of the spherical surface with \( r = b_r \).

A solid sphere with radius \( R \) and center at the origin is an example of a spherical box with \( a_r = 0, b_r = R, a_\theta = 0, b_\theta = \pi, a_\phi = 0, \) and \( b_\phi = 2\pi \). A hemisphere with the same radius and center, located on and above the plane \( z = 0 \) is a spherical box with the same limits except that \( b_\theta = \pi/2 \).

The tiny volumes \( \Delta V \) that are used to define volume integrals in spherical coordinates are tiny spherical boxes as illustrated in Figure A.6. As \( \Delta V \) tends to zero, its shape approaches a rectangular box with length \( L = \Delta r \), depth \( D \), and height \( H \), as indicated in the figure. The depth is the length of a circular arc with angle \( \Delta \phi \) and radius equal to the distance of \( \Delta V \) from the \( z \) axis, which is \( r \sin \theta \). Hence \( D = r \sin \theta \Delta \phi \). The height is the length of a circular arc with
angle $\Delta \theta$ and radius equal to $r$. Hence, $H = r\Delta \theta$. Consequently, the volume of the rectangular box approaches $LHD = r^2 \sin \theta \Delta r \Delta \theta \Delta \phi$, which yields the element of volume given by\(^6\)

$$dV = r^2 \sin \theta \, dr \, d\theta \, d\phi.$$  \hspace{1cm} (A.26)

![Figure A.5: Spherical box.](image)

![Figure A.6: The elemental volume in spherical coordinates.](image)

For the volume integral of the spherical box defined in Eqns. (A.23) - (A.25), the limits of integration are obvious. Hence, with the element of volume given by Eqn. (A.26), our volume integral becomes

$$I = \int_{a_r}^{b_r} \int_{a\theta}^{b\theta} \int_{a\phi}^{b\phi} f(r, \theta, z) \, r^2 \sin \theta \, dr \, d\theta \, d\phi.$$  \hspace{1cm} (A.27)

This is a triple iterated integral, in which we first evaluate the definite $r$ integral, followed by evaluation of the definite $\theta$ integral, and completed by evaluation of the definite $\phi$ integral. Since none of the limits of integration depend on the variables of integration, we can interchange the orders of integration any way we wish without complication. The six possibilities for $dV$ are $(r^2 \sin \theta \, dr \, d\theta \, d\phi)$, $(r^2 \sin \theta \, d\theta \, d\phi \, dr)$, $(r^2 \sin \theta \, d\phi \, dr \, d\theta)$, $(r^2 \sin \theta \, dr \, d\phi \, d\theta)$, $(r^2 \sin \theta \, d\theta \, dr \, d\phi)$, and $(r^2 \sin \theta \, d\phi \, d\theta \, dr)$. When interchanging the order of integration, be sure to switch the limits of integration as well so that the limits of the $r$ integral are always $a_r$ and $b_r$ and similarly for the limits of the $\theta$ and $\phi$ integrals.\(^7\)

**Sample Problem A.2** A sphere of radius $R$ centered at the origin has volume charge density $\rho(r, \theta, \phi) = K \cos \theta \, r^3$ where $R$ and $K$ are positive constants. Determine the total charge in the portion of the sphere located in the first octant.

\(^6\)When specifying the limits of integration, it is important to keep the element of volume $dV$ positive. One way to do this is to choose each of the three integrations to be in the positive direction and keep $\theta$ in the range $0 \leq \theta \leq \pi$.

\(^7\)To make sure that you understand how to apply Eqn. (A.27) to simple geometries, use it to solve Sample Problem A.1.
Solution:

Givens: \( R, K \).

The volume of interest \( V \) is a spherical box with \( r \) ranging from 0 to \( R \), \( \theta \) ranging from 0 to \( \pi/2 \), and \( \phi \) ranging from 0 to \( \pi/2 \). Hence, the total charge in \( V \) is

\[
Q = \int_V \rho(r, \theta, \phi) \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^R K \cos \theta \, r^3 r^2 \sin \theta \, dr \, d\theta \, d\phi.
\]

Since the \( \phi \) integral is trivial, let’s do it first.

\[
Q = K \int_0^{\pi/2} \int_0^R \int_0^{\pi/2} r^5 \cos \theta \, \sin \theta \, d\phi \, dr \, d\theta = K \int_0^{\pi/2} \int_0^R r^5 \cos \theta \, \sin \theta \, dr \, d\theta.
\]

Next, we evaluate the \( r \) integral

\[
Q = K \frac{\pi}{2} \frac{R^6}{6} \int_0^{\pi/2} \cos \theta \, \sin \theta \, d\theta.
\]

Finally, we do the \( \theta \) integral using the \( u \)-substitution \( u = \sin \theta \) with \( du = \cos \theta \, d\theta \) to obtain

\[
Q = K \frac{\pi}{2} \frac{R^6}{6} \frac{1}{2} = \frac{\pi}{24} KR^6.
\]
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