A sequence is an unending succession of numbers, called terms. It is understood that the terms have a definite order.

\[ a_1, a_2, a_3, \ldots \]

**Ex**

(a) 1, 2, 3, 4, \ldots  
(b) \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \)  
(c) 2, 4, 6, 8, \ldots  
(d) 1, 1, 1, \ldots

It is better to have a rule or formula for generation of terms. One way of doing this is to look for a function that relates each term in the sequence to its term number.

**Ex**

(a) 2, 4, 6, 8, \ldots, 2n, \ldots \quad f(n) = 2n

(b) \( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots \) \quad f(n) = \frac{n}{n+1}

(c) \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots \) \quad f(n) = \frac{1}{2^n}

(d) \( \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1} \frac{n}{n+1}, \ldots \) \quad f(n) = (-1)^{n+1} \frac{n}{n+1}
When the general term of a sequence \( a_1, a_2, a_3, \ldots, a_n, \ldots \) is known, we only write the general term in braces:

\[
\{ a_n \}_{n=1}^{+\infty}
\]

**Example:** \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{+\infty}, \left\{ \frac{1}{2^n} \right\}_{n=1}^{+\infty}, \left\{ \frac{1}{2^n-1} \right\}_{n=1}^{+\infty}, \left\{ \frac{1}{2^n} \right\}_{n=0}^{+\infty} \)

**Definition:** A sequence is a function whose domain is a set of integers.

**Limit of a sequence:**

- \( \left\{ n+1 \right\}_{n=1}^{+\infty} \): increases without bound
- \( \left\{ (-1)^{n+1} \right\}_{n=1}^{+\infty} \): oscillates between 1 and -1
- \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{+\infty} \): increases toward a "limiting value" of 1.
tends toward a "limiting value" of 1, but do so in an oscillation fashion.

\[ \left\{ 1 + \left( -\frac{1}{2} \right)^n \right\}_{n=1}^{+\infty} \]

**Definition** A sequence \( \{a_n\} \) is said to converge to the limit \( L \) if given any \( \epsilon > 0 \), there is a positive integer \( N \) such that \( |a_n - L| < \epsilon \) for \( n \geq N \). In this case we write

\[ \lim_{n \to +\infty} a_n = L \]

A sequence that does not converge to some finite limit is said to diverge.

From this point on, the terms in the sequence are all within \( \epsilon \) units of \( L \).
**THEOREM**  Suppose that the sequences \( \{a_n\} \) and \( \{b_n\} \) converge to limits \( L_1 \) and \( L_2 \), respectively, and \( c \) is a constant. Then:

(a) \[ \lim_{n \to +\infty} c = c \]

(b) \[ \lim_{n \to +\infty} c a_n = c \lim_{n \to +\infty} a_n = c L_1 \]

(c) \[ \lim_{n \to +\infty} (a_n + b_n) = \lim_{n \to +\infty} a_n + \lim_{n \to +\infty} b_n = L_1 + L_2 \]

(d) \[ \lim_{n \to +\infty} (a_n - b_n) = \lim_{n \to +\infty} a_n - \lim_{n \to +\infty} b_n = L_1 - L_2 \]

(e) \[ \lim_{n \to +\infty} (a_n b_n) = \lim_{n \to +\infty} a_n \cdot \lim_{n \to +\infty} b_n = L_1 L_2 \]

(f) \[ \lim_{n \to +\infty} \left( \frac{a_n}{b_n} \right) = \frac{\lim_{n \to +\infty} a_n}{\lim_{n \to +\infty} b_n} = \frac{L_1}{L_2} \quad (\text{if } L_2 \neq 0) \]
If \( f(x) \to L \) as \( x \to +\infty \), then \( f(n) \to L \) as \( n \to +\infty \).

\[
\begin{align*}
\text{Ex} \quad \lim_{n \to \infty} \frac{n}{2n+1} &= \frac{1}{2} \\
\text{L’Hôpital.}
\end{align*}
\]

\[
\begin{align*}
\text{Ex} \quad \lim_{n \to \infty} \frac{(\frac{1}{2})^n}{n} &= 0 \\
\text{oscillates between 1 and -1, diverges}
\end{align*}
\]

\[
\begin{align*}
\text{Ex} \quad \lim_{n \to \infty} (8 - 2n) &= -\infty, \text{ diverges}
\end{align*}
\]

\[
\begin{align*}
\text{Ex}\quad \text{Does the following sequences converge or not?}
\end{align*}
\]

\[
(a) \quad 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots \quad \{\frac{1}{2^n}\}_{n=1}^{+\infty}, \quad \lim_{n \to \infty} \frac{1}{2^n} = 0
\]
(b) \( 1, 2, 2^2, 2^3, \ldots, 2^n, \ldots \) \( \lim_{n \to \infty} 2^n = \infty \), seq. \( \{2^n\} \) diverges.

Ex. Find the limit of the sequence \( \left\{ \frac{n}{e^n} \right\}_{n=1}^{+\infty} \)

\[
\lim_{n \to \infty} \frac{n}{e^n} = \lim_{n \to \infty} \frac{1}{e^n} = 0 \quad \text{L'Hôpital.}
\]

Ex. Show that \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \)

\[
y = \sqrt[n]{n} \Rightarrow \lim_{n \to \infty} \ln y = \lim_{n \to \infty} \frac{1}{n} \ln n = \lim_{n \to \infty} \frac{\ln n}{n}
\]

\[
\lim_{n \to \infty} \ln y = \lim_{n \to \infty} \frac{1/n}{1} = 0 \quad \text{L'Hôpital.}
\]

\[
\lim_{n \to \infty} \ln y = 0
\]

\[
\lim_{n \to \infty} y = e^0 = 1
\]

\[
\lim_{n \to \infty} \sqrt[n]{n} = 1
\]
Ex \( \left\{ \left( 1 + \frac{1}{n} \right)^n \right\}_{n=1}^{+\infty} \), does this sequence converge or not?

\[ y = \left( 1 + \frac{1}{n} \right)^n \Rightarrow \ln y = n \cdot \ln \left( 1 + \frac{1}{n} \right) = \frac{\ln \left( 1 + \frac{1}{n} \right)}{1/n} \]

\[ \lim_{n \to \infty} \ln y = \lim_{n \to \infty} \frac{\ln \left( 1 + \frac{1}{n} \right)}{1/n} \]

\[ = \lim_{n \to \infty} \frac{-1/n}{1 + 1/n} = 1 \]

\[ \lim_{n \to \infty} \ln y = 1 \]

\[ \lim_{n \to \infty} y = e \]

\[ \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e \], the sequence converges to \( e \).
**Theorem (The Sandwich Theorem for Sequences)** Let \( \{a_n\} \), \( \{b_n\} \), and \( \{c_n\} \) be sequences such that

\[
a_n \leq b_n \leq c_n \quad \text{for all values of } n \text{ beyond some index } N
\]

If the sequences \( \{a_n\} \) and \( \{c_n\} \) have a common limit \( L \) as \( n \to +\infty \), then \( \{b_n\} \) also has the limit \( L \) as \( n \to +\infty \).

If \( a_n \to L \) and \( c_n \to L \), then \( b_n \to L \).
Ex. Find the limit of \( \left\{ \frac{\sin(n!)}{n^2} \right\}_{n=1}^{+\infty} \)

\[
\frac{(n+1)!}{n^2} \leq \frac{\sin(n!)}{n^2} \leq 1 \cdot \frac{(n+1)}{n^2}
\]

\[
-\frac{n+1}{n^2} \leq \frac{\sin n!}{n^2} \cdot (n+1) \leq \frac{n+1}{n^2}
\]

\[
\lim_{n \to \infty} -\left(\frac{1}{n} + \frac{1}{n^2}\right) \leq \frac{\sin n!}{n^2} \cdot (n+1) \leq \left(\frac{1}{n} + \frac{1}{n^2}\right)
\]

\[
\lim_{n \to \infty} \frac{\sin n!}{n^2} = 0
\]

HW \( \left\{ \cos \left(\frac{n}{n}\right) \right\}, \left\{ \frac{1}{2^n} \right\}, \left\{ (-1)^n \frac{1}{n} \right\} \) find their limits.
Sequences defined recursively

\[ x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right), \quad x_1 = 1 \]

\[ x_2 = \frac{1}{2} \left( x_1 + \frac{2}{x_1} \right) = \frac{1}{2} \left( 1 + \frac{2}{1} \right) = \frac{3}{2} \]

\[ x_3 = \frac{1}{2} \left( x_2 + \frac{2}{x_2} \right) = \frac{1}{2} \left( \frac{3}{2} + \frac{2}{\frac{3}{2}} \right) \]

Monotone Sequences

<table>
<thead>
<tr>
<th>SEQUENCE</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots )</td>
<td>Strictly increasing</td>
</tr>
<tr>
<td>1, ( \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} ), \ldots</td>
<td>Strictly decreasing</td>
</tr>
<tr>
<td>1, 1, 2, 2, 3, 3, \ldots</td>
<td>Increasing; not strictly increasing</td>
</tr>
<tr>
<td>1, 1, ( \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3} ), \ldots</td>
<td>Decreasing; not strictly decreasing</td>
</tr>
<tr>
<td>1, (-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1} \frac{1}{n}, \ldots )</td>
<td>Neither increasing nor decreasing</td>
</tr>
</tbody>
</table>
\[
\left\{ \frac{n}{n+1} \right\}_{n=1}^{+\infty} \\
\left\{ \frac{1}{n} \right\}_{n=1}^{+\infty} \\
1, 1, 2, 2, 3, 3, \ldots \\
1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots \\
\left\{ (-1)^{n+1} \frac{1}{n} \right\}_{n=1}^{+\infty}
\]
<table>
<thead>
<tr>
<th>Difference Between Successive Terms</th>
<th>Ratio of Successive Terms</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{n+1} - a_n &gt; 0$</td>
<td>$a_{n+1}/a_n &gt; 1$</td>
<td>Strictly increasing</td>
</tr>
<tr>
<td>$a_{n+1} - a_n &lt; 0$</td>
<td>$a_{n+1}/a_n &lt; 1$</td>
<td>Strictly decreasing</td>
</tr>
<tr>
<td>$a_{n+1} - a_n \geq 0$</td>
<td>$a_{n+1}/a_n \geq 1$</td>
<td>Increasing</td>
</tr>
<tr>
<td>$a_{n+1} - a_n \leq 0$</td>
<td>$a_{n+1}/a_n \leq 1$</td>
<td>Decreasing</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Derivative of $f$ for $x \geq 1$</th>
<th>Conclusion for the Sequence with $a_n = f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x) &gt; 0$</td>
<td>Strictly increasing</td>
</tr>
<tr>
<td>$f'(x) &lt; 0$</td>
<td>Strictly decreasing</td>
</tr>
<tr>
<td>$f'(x) \geq 0$</td>
<td>Increasing</td>
</tr>
<tr>
<td>$f'(x) \leq 0$</td>
<td>Decreasing</td>
</tr>
</tbody>
</table>
**THEOREM** If a sequence \( \{a_n\} \) is eventually increasing, then there are two possibilities:

(a) There is a constant \( M \), called an **upper bound** for the sequence, such that \( a_n \leq M \) for all \( n \), in which case the sequence converges to a limit \( L \) satisfying \( L \leq M \).

(b) No upper bound exists, in which case \( \lim_{n \to +\infty} a_n = +\infty \).

**THEOREM** If a sequence \( \{a_n\} \) is eventually decreasing, then there are two possibilities:

(a) There is a constant \( M \), called a **lower bound** for the sequence, such that \( a_n \geq M \) for all \( n \), in which case the sequence converges to a limit \( L \) satisfying \( L \geq M \).

(b) No lower bound exists, in which case \( \lim_{n \to +\infty} a_n = -\infty \).

**Ex.** Show that \( \left\{ \frac{10^n}{n!} \right\}_{n=1}^{+\infty} \) converges and find its limit.

\[
\begin{align*}
 a_n &= \frac{10^n}{n!} \\
 a_{n+1} &= \frac{10^{n+1}}{(n+1)!} \\
 \Rightarrow \frac{a_{n+1}}{a_n} &= \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \frac{10}{n+1} < 1 \\
 a_{n+1} &= \frac{10^{n+1}}{(n+1)!} = \frac{10}{n+1} \cdot \left( \frac{10^n}{n!} \right) = \frac{10}{n+1} a_n
\end{align*}
\]
\[\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{10}{n+1} a_n \]
\[= \lim_{n \to \infty} \frac{10}{n+1} \lim_{n \to \infty} a_n = 0 \]

\[L = \lim_{n \to \infty} \frac{10^n}{n!} = 0 \]

\[\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n = L \quad \text{THM.} \]

For any real value of \( x \),
\[\lim_{n \to \infty} \frac{x^n}{n!} = 0 \]

Ex. Does \( \left\{ \frac{n-1}{n} \right\} \) converge? Limit?

Let us use calculus: \( f(x) = (1 - \frac{1}{x}) \Rightarrow f'(x) = \frac{1}{x^2} > 0 \), \( x > 1 \)
\( f(x) \) monotonically increases, so does the sequence. \( \lim_{x \to \infty} (1 - \frac{1}{x}) = 1 \)