Integral Test

The area under curve is $y = f(x)$, which is an improper integral and the series may approximate this area.

$$\text{Area} = \int_{1}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{1}^{b} f(x) \, dx$$

In terms of rectangles, if we use $n$ rect:

$$\int_{1}^{n+1} f(x) \, dx \leq \sum_{k=1}^{n} a_k \quad \text{(for Fig. a)}$$

If we use Fig. b: $a_1 + a_2 + \ldots \leq \int_{1}^{n+1} f(x) \, dx$

So,

$$\sum_{k=2}^{n+1} a_k \leq \int_{1}^{n+1} f(x) \, dx \leq \sum_{k=1}^{n} a_k$$

$\lim_{n \to \infty} b_n < \infty$ implies that $\lim_{n \to \infty} a_n < \infty$, so that series must converge.
The p-Series

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} \text{, for what values of } p \text{, does it converge?} \]

\[ \int_{1}^{\infty} \frac{1}{x^p} \, dx = \lim_{b \to \infty} \int_{1}^{b} x^{-p} \, dx = \left. \frac{x^{-p+1}}{-p+1} \right|_{1}^{b}, \text{ if } p > 1, \text{ it converges; diverges otherwise.} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} \text{, since } p=1, \text{ it diverges.} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n^2+1} \]

\[ \int_{1}^{\infty} \frac{1}{x^2+1} \, dx = \lim_{b \to \infty} \arctan x \bigg|_{1}^{b} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \]
\[ \int \frac{\tan^{-1} x}{1 + x^2} \, dx = \tan^{-1} x \cdot \frac{1}{1 + x^2} \, dx = \int \frac{u}{1 + u^2} \, du \]

where \( u = \tan^{-1} x \).

Using the substitution \( u = \tan^{-1} x \), we have \( du = \frac{1}{1 + x^2} \, dx \), so the integral becomes

\[ \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} \tan^{-1}^2 x + C. \]

The series diverges by the integral test.

\[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{3}{32} \pi^2. \]
We have to have a nice collection of known convergent series to compare an unknown series to.

**Example**

\[
\sum_{n=1}^{\infty} \frac{1}{n!} \quad \text{for all } n.
\]

\[
\sum_{n=1}^{\infty} \frac{1}{2^n}
\]

converges to zero, so does \( \sum_{n=1}^{\infty} \frac{1}{n!} \).

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**THEOREM** — The Comparison Test

Let \( \sum a_n, \sum c_n, \text{ and } \sum d_n \) be series with nonnegative terms. Suppose that for some integer \( N \)

\[
d_n \leq a_n \leq c_n \quad \text{for all } n > N.
\]

(a) If \( \sum c_n \) converges, then \( \sum a_n \) also converges.

(b) If \( \sum d_n \) diverges, then \( \sum a_n \) also diverges.

**Example**

\[
\sum_{n=2}^{\infty} \frac{1}{\ln n}, \text{ we know that } \sum \frac{1}{n}
\]

\[
\frac{1}{\ln n} \geq \frac{1}{n}
\]

\[
\sum_{n=2}^{\infty} \frac{1}{\ln n} \text{ is divergent.}
\]
\[ \sum \frac{1}{n(n+1)}, \text{ is it convergent? It is similar to } \sum \frac{1}{n^2}, \text{ is conv.} \]

However, \( \frac{1}{n^2} > \frac{1}{n(n+1)} \), therefore we need to apply the following second test.

**THEOREM** — Limit Comparison Test

Suppose that \( a_n > 0 \) and \( b_n > 0 \) for all \( n \geq N \) (\( N \) an integer).

1. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = c > 0 \), then \( \sum a_n \) and \( \sum b_n \) both converge or both diverge.

2. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \) and \( \sum b_n \) converges, then \( \sum a_n \) converges.

3. If \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \) and \( \sum b_n \) diverges, then \( \sum a_n \) diverges.

\[
\lim_{n \to \infty} \frac{1/n^2}{1/n(n+1)} = \lim_{n \to \infty} \frac{n^2 + n}{n^2} = \lim_{n \to \infty} \left(1 + \frac{1}{n^2}\right) = 1
\]
Ex \[ \sum_{n=1}^{\infty} \frac{1}{n^{1/2} + 3n^{1/2} - 5} \] is it convergent or not?

\[ \ln \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \], since \( p = 1/2 < 1 \), it is divergent

We can compare with \( \sum \frac{1}{n^{1/2}} \)

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{1/n^{1/2}}{n^{1/2} + 3n^{1/2} - 5} = \lim_{n \to \infty} \left( 1 + \frac{3}{n^{1/2}} - \frac{5}{n^{1/2}} \right) = 1 \]

\[ \therefore \sum_{n=1}^{\infty} \frac{1}{n^{1/2} + 3n^{1/2} - 5} \] is divergent!