10.7 Power Series

**DEFINITIONS** A power series about \( x = 0 \) is a series of the form

\[
\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \tag{1}
\]

A power series about \( x = a \) is a series of the form

\[
\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n + \cdots \tag{2}
\]

in which the center \( a \) and the coefficients \( c_0, c_1, c_2, \ldots, c_n, \ldots \) are constants.

**Ex** \[
\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}, \quad |x| < 1 \Rightarrow -1 < x < 1
\]

The graphs of \( f(x) = 1/(1 - x) \) in Example and four of its polynomial approximations.
\[ E_x \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n (x-2)^n = 1 - \frac{1}{2} (x-2) + \frac{1}{4} (x-2)^2 - \ldots = \frac{1}{1 + \frac{x-2}{2}} \]

| - \frac{x-2}{2} | < 1 \Rightarrow x > 0, \ x < 4 \quad = \frac{2}{x}

For \( x = 0 \), \( \sum_{n=0}^{\infty} 2^n \), diverges.

For \( x = 4 \), \( \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \ldots \), diverges.

The graphs of \( f(x) = \frac{2}{x} \)
and its first three polynomial approximations

Generalized Ratio Test

Let \( \sum a_n \) be any series, let \( p = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \)

(i) the series converges absolutely if \( p < 1 \).
(ii) the series diverges if \( p > 1 \).
(iii) the test gives no information if \( p = 1 \).
Ex. Find all the values of $x$ for which the series
\[ \sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{2^n n^2} \] is convergent.

\[ p = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(x+1)^{n+1}}{2^{n+1}(n+1)^2}}{\frac{x+1}{2^n n^2}} \right| = \lim_{n \to \infty} \frac{1}{2} \left| x+1 \right| \left( \frac{n}{n+1} \right)^2 \]

\[ p = \frac{1}{2} \left| x+1 \right| \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^2 = \frac{1}{2} \left| x+1 \right| \lim_{n \to \infty} \left( \frac{1}{1+\frac{1}{n}} \right)^2 = \frac{1}{2} \left| x+1 \right| \]

The series converges absolutely if
\[ \frac{1}{2} \left| x+1 \right| < 1 \Rightarrow -3 < x < 1 \]

Now, we must test the end pts $x=1$ and $x=-3$:

For $x=1$
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \], converges absolutely, since it is a $p$-series with $p=2$.

For $x=-3$
\[ \sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \], converges absolutely. \[ \therefore -3 \leq x \leq 1 \]
The set of all $x$ for which a power series is convergent is called the **interval of convergence**. Notice that for a power series of the type $\sum c_n (x-a)^n$ the ratio of two consecutive terms will always contain a term like $|x-a|$, and $r = |x-a|$ something.

If the something is positive real number, then the series converges on an interval. If the something is zero, the series has ratio $r = 0$ for all $x$, so converges everywhere. If the something is $\infty$, the series diverges everywhere except at $x=a$, where the series collapses to $a_0$. This consideration gives the following theorem:

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**THEOREM**  

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number $R$ such that the series diverges for $x$ with $|x-a| > R$ but converges absolutely for $x$ with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every $x$ ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

$R$ is called the **radius of convergence** of the power series.
\[ \sum_{n=0}^{\infty} \frac{(-1)^n n! x^n}{10^n} \]

\[ P = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n \cdot 10^{n+1}} \frac{n! x^n}{10^n} \right| = \frac{|x|}{10} \lim_{n \to \infty} \frac{n!}{n+1} = \infty, \]

unless \( x = 0 \)

The series converges only at \( x = 0 \).

\[ \sum_{n=1}^{\infty} \frac{x^n}{n!} \]

\[ P = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n! x^n}{n! x^n} \right| = |x| \lim_{n \to \infty} \frac{1}{n+1} = 0 \quad \text{for all } x. \]

The series converges for all \( x \).
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} \]

\[ p = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-1| \lim_{n \to \infty} \frac{n}{n+1} = |x-1| \]

The series converges if \(|x-1| < 1 \Rightarrow 0 < x < 2\).

For \(x = 0\): \( \sum -\frac{1}{n} = -\sum \frac{1}{n} \), diverges

For \(x = 2\): \( \sum (-1)^n \), known to be convergent

\[ \sum_{n=1}^{\infty} \frac{\arctan n}{n} \], we shall use comparison test

\[ \lim_{n \to \infty} \frac{\frac{\arctan n}{n}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{\arctan n}{n} = \frac{\pi}{2} \]

\( \sum \frac{\arctan n}{n} \) and \( \sum \frac{1}{n} \) are comparable. Since \( \sum \frac{1}{n} \) is divergent, \( \sum \frac{\arctan n}{n} \) is also divergent.
Ex. \( \sum_{n=2}^{\infty} \frac{1}{n \ln(n)^s} \) we shall use the integral test

\[
\int_{2}^{\infty} \frac{dx}{x \ln(x)^s} = \left[ \frac{\ln(x)^{1-s}}{1-s} \right]_{2}^{\infty} = \begin{cases} 
\text{Div.} & 1-s > 0 \\
\text{Conv.} & 1-s < 0 
\end{cases}
\]

Thus the series diverges for \( s < 1 \), converges for \( s > 1 \); this is for positive \( s \).

HW: study if \( s \leq 0 \) (it diverges)