Vector-valued functions
Also called vector functions. It is a function of one or more variables whose range is in vectors. \( \mathbf{r}(t) = a(t)i + b(t)j + c(t)k \)

Ex: In 2D, the position vector as a function of time: \( \mathbf{r}(t) = x(t)i + y(t)j \)
The derivative can be defined simply as:
\[
\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}
\]
The velocity vector:
\[
\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j
\]
acceleration:
\[
\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j
\]
Note that the \( \mathbf{v}(t) \) vector is tangent to the trajectory of the object.
\[
\lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \mathbf{v}(t)
\]
For simple projectile motion:
\[
\mathbf{r}(t) = (v_i \cos \theta t + x_0)i + (-\frac{1}{2}gt^2 + v_i \sin \theta t + y_0)j
\]
\[
\mathbf{v}(t) = v_i \cos \theta i + (-gt + v_i \sin \theta)j
\]
Acceleration vector:
\[
\mathbf{a}(t) = 0i + (-g)j
\]
(please note that while taking the derivative, we took \( \frac{dx}{dt} = 0 \) and \( \frac{dy}{dt} = 0 \), i.e. we took \( i \) and \( j \) to be independent of time. If the coordinate system itself were to move over time, we would need to handle them as well.)

Having defined the position as a vector, we can also define average velocity as a vector as well. Imagine a particle moving apparently randomly, but with an overall displacement. Consider for example a ball falling through a set of pegs, we can talk about a net displacement. (This is similar to electrons moving in a conducting wire under an electric field)

\[
\Delta \mathbf{r} = \mathbf{r}_f - \mathbf{r}_i \quad \text{displacement vector}
\]
\[
\mathbf{v}_{avg} = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}_f - \mathbf{r}_i}{\Delta t} \quad \text{Instantaneous velocity we have defined earlier is the limit of} \quad \frac{\Delta \mathbf{r}}{\Delta t} \quad \text{as} \quad \Delta t \to 0.
\]

Likewise if we know that the velocity changes over time, we can define an average acceleration, i.e. \( \mathbf{a}_{avg} = \frac{\mathbf{v}_f - \mathbf{v}_i}{\Delta t} = \frac{\Delta \mathbf{v}}{\Delta t} \)

An excellent application of defining the position as a vector is the study of uniform circular motion. Motion on a circular trajectory with constant speed.
The amount of angle covered per unit time is independent of time.
Let's start by putting the origin of our Cartesian coordinate system at the center of the circle.
\[
\mathbf{r}(t) = r \cos \theta(t)i + r \sin \theta(t)j
\]
\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} = r(-\sin \theta) \frac{d\theta}{dt} i + r \cos \theta \frac{d\theta}{dt} j = rw \sin \theta \frac{d\theta}{dt}
\]
\[
\mathbf{a} = \frac{d\mathbf{v}}{dt} = r(-\cos \theta) \frac{d^2 \theta}{dt^2} i - r \sin \theta \frac{d^2 \theta}{dt^2} j = rw \cos \theta \frac{d^2 \theta}{dt^2}
\]
where we defined \( \omega = \frac{d\theta}{dt} \), constant in u.m.
And we made use of the chain rule for derivatives: \( f(g(x))' = f'(g(x)) \cdot g'(x) \).
\[
\mathbf{v} = xw^2 - y \omega = \omega(x^2 - y^2)
\]
\[
\text{speed} = |\mathbf{v}| = \sqrt{x^2 + (y^2)} = \omega \cdot r = \nu \quad \Rightarrow \quad \omega^2 = \frac{\nu^2}{r^2}
\]
We also note that the slope of \( \mathbf{v} \) is the negative reciprocal of the slope of \( \mathbf{r} \). Hence they are perpendicular: \( \mathbf{v} \cdot \mathbf{r} = 0 \)

[ A small math exercise: show that for any two perpendicular lines, the product of the slopes is equal to -1.]

[How about the acceleration?
\[
\mathbf{a} = \frac{d^2 \mathbf{r}}{dt^2} = r \omega (-\sin \theta) \frac{d^2 \theta}{dt^2} i - r \omega \cos \theta \frac{d^2 \theta}{dt^2} j = -rw^2 (\sin \theta \dot{\omega} + \omega \ddot{\theta}) \]
\[
\mathbf{a} = (\omega^2 \mathbf{r}) = \text{acceleration in u.m. is always antiparallel to the position vector, i.e. always towards the center of the circle.}
\]
\[
|\mathbf{a}| = \omega^2 r = \frac{\nu^2}{r} \quad \text{period of time spent for one complete rotation}
\]
\[
T = \frac{2\pi}{\nu}
\]
\[
\omega = \frac{2\pi}{T}
\]