We note that this formula applies for \( \theta > \pi/2 \) as well as for \( \theta < \pi/2 \). When \( \theta > \pi/2 \), \( \cos \theta < 0 \) and hence \( h > R \).

\[ \text{Going back to our question, what height does the car lose contact with the rail? This is the same as asking for what height does } N \text{ become zero? Let us call that height } h_{\text{No}}. \text{ When } h = h_{\text{No}}: \]

\[ N = 0 \Rightarrow -mg\cos \theta = \frac{mv^2}{R} \]

\[ v^2 = 2(g(h_i - h_{\text{No}})) = 2g(h_i - R + R\cos \theta_{\text{No}}) \]

\[ -g\cos \theta = \frac{2g}{R} \cdot (h_i - R + R\cos \theta_{\text{No}}) \]

\[ \frac{3}{2} \cdot \cos \theta_{\text{No}} = 1 - \frac{h_i}{R} \Rightarrow \cos \theta_{\text{No}} = \frac{2}{3} \left(1 - \frac{h_i}{R}\right) \]

Hence, \( h_{\text{No}} = R \left(1 - \frac{2}{3} + \frac{h_i}{3R}\right) = R \left(\frac{1}{3} + \frac{2h_i}{3R}\right) \)

We note that for \( h_i = R \), \( h_{\text{No}} = R \), as we would expect. The “surprise” is the fact that for \( h_i > R \), \( h_{\text{No}} < h_i \):

\[ h_{\text{No}} = h_i \left(\frac{R}{3h_i} + \frac{2}{3}\right) < h_i \text{ (as } \frac{R}{h_i} < 1\text{)} \]

Therefore, if we start with, say \( h_i = \frac{4}{3}R \), or even with \( h_i = 2R \), our car would not be able to complete the loop. Instead, somewhere above the height of \( R \), it would start doing a simple projectile motion in air.

What is the minimum \( h_i \) for which we will not leave the rail?

For this we want \( h_{\text{No}} > 2R \)

\[ h_i \left(\frac{R}{3h_i} + \frac{2}{3}\right) > 2R \Rightarrow h_i > \frac{5}{2}R \]

What is the highest normal force that the rail applies?

For \( \theta = 0 \Rightarrow N = m \frac{v^2}{R} = \frac{2mg h_i}{R} + mg \cos \theta = 0 \)

For the minimum safe \( h_i \):

\[ N_m = \frac{2mg \frac{2R}{3}}{R + mg} \]

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*Ex: Amusement park ride - rather famous problem (inspired by Sernay 8:72). The roller-coaster track has a circular, radius R, loop with height \( h_i \). If the roller-coaster car is released from an initial height \( h_i \), describe what happens for various magnitudes of \( h_i \). Neglect all friction.

Case 1: \( h_i < R \). The max. height to be reached is equal to \( h_i \).

Case 2: \( h_i > R \). Is the max height to be reached still \( h_i \)? For this we need to answer the question: Does the car lose contact with the rail as it goes above the height of \( R \)? If the speed is small enough, the centripetal acceleration can be provided by the gravity and the need for a rail can be lost.

For what values of \( h_i \) are we safe? Or at what height in the loop do we lose contact with the rail?

The component of \( mg \) that is tangential to the loop decreases the speed of the car. Its radial component is the normal force for the radial acceleration.

\[ N - mg\cos \theta = F_{\text{Radial}} = m \cdot \frac{v^2}{R} \]

Note that for \( 0 < \theta < \pi/2 \), \( mg \) is opposing the normal force, while for \( \pi/2 < \theta < 3\pi/2 \), it is in the same direction.

From the conservation of energy, when the car is some height \( h \), irrespective of whether it is or is not on the track, we can write:

\[ mg h_i = \frac{1}{2}mv^2 + mgh \]

If the car is on the track, from geometry we can write: \( h = R - R\cos \theta = R(1 - \cos \theta) \)