A physicist's introduction to integrals

Let us start with the following problem. Imagine a bar made up of two different materials, like silver and gold.

\[ \rho_{\text{m}} = 19.3 \text{ g/cm}^3 \quad \rho_{\text{Ag}} = 10.5 \text{ g/cm}^3 \]

Say the bar has cross-section area \( A \) and length \( L \).

What is the total mass?

\[ m = m_1 + m_2 \]

where

\[ m_1 = \rho_{\text{m}} \cdot A \cdot \text{index} \]

\[ m_2 = \rho_{\text{Ag}} \cdot \text{mass of the silver part} \]

But each piece we label in some way. Either by color, or better yet with a unique index, like \( i = 1, i = 2 \).

An even better approximation is given by dividing the bar into \( n \) pieces.

\[ m = \sum_{i=1}^{n} p_i \cdot V_i = \sum_{i=1}^{n} \rho_i \cdot A \cdot \text{length of each piece} \]

Now let us look at a more general problem. We heat up the bar so the gold and silver atoms diffuse and mix up.

The concentration of silver starts at 100% at the left end and gradually goes down to 0% at the right end.

What is the mass? (If we added no material, it should still be the same as before, but let us try to compute this.)

At a very bad approximation, we can assume that the whole thing is made of silver, i.e. we assume that the concentration of silver is everywhere the same as the concentration at the left end (i.e., 100%).

Then:

\[ m \approx \rho_{\text{Ag}} \cdot V \]

A better approximation can be obtained by dividing the bar into two and assuming that the first half has the concentration of the left end and the second half has the concentration of the middle point.

\[ \frac{1}{2} \rho_{\text{Ag}} + \frac{1}{2} \rho_{\text{m}} \]

approx 1: \[ m \approx \rho_{\text{Ag}} \cdot V \]

approx 2: \[ m = m(1) + m(2) = \rho_1 \cdot V_1 + \rho_2 \cdot V_2 = \rho_{\text{Ag}} \cdot \frac{V_1}{2} \]

\[ \rho_{\text{Ag}} \downarrow \rho_{\text{Ag}} + \rho_{\text{m}} \]

approx 3: \[ m = \sum_{i=1}^{4} p_i \cdot V_i = \frac{\rho_1 \cdot V_1}{4} + \frac{(0.75 \rho_{\text{Ag}} + 0.25 \rho_{\text{m}}) \cdot V_2}{4} + \frac{(0.25 \rho_{\text{Ag}} + 0.75 \rho_{\text{m}}) \cdot V_3}{4} + \frac{\rho_3 \cdot V_4}{4} \]

We would get a "perfect" approximation if we could divide the object to \( N \) equal parts and choose \( N \) as a very large number, \( N \to \infty \).

But as we take \( N \) to infinity, the \( i \)th piece will change location. With \( N = 4 \), the 3rd piece sits at halfway thru the bar, whose as for \( N = 100 \), the 3rd piece lies very close to the left end of the rod. This makes defining \( p(i) \), etc. difficult.

So let us not use an integer index but instead some real number as a label.
The distance between two consecutive pieces is: \( \Delta x = \frac{x_i - x_{i-1}}{n} \). As \( n \to \infty \), the width of each piece approaches 0, and the sum becomes the definite integral. 

The riemann sum is: 

\[
\sum_{i=1}^{n} f(x_i) \Delta x
\]

The definite integral is: 

\[
\int_{a}^{b} f(x) \, dx
\]

The definite integral is the limit of riemann sums as the number of subintervals approaches infinity.