Consider we are flying from some point A to point B. The wind will be blowing at different strengths in different directions at different locations. So the force that the wind is going to apply at every point of our flight path will be different. Now consider different flight routes, which will need the minimum fuel? Or we need to compute the total work done by the wind over the whole path.

Ex: Imagine the force at any point on the x-y plane is given by the function:

\[ F(x,y) = \alpha x y + \beta x \hat{y} \]

So, you give me your coordinates, I tell you the force of the wind at that location. Examples:

- At the origin: \((x,y) = (0,0) \Rightarrow F(0,0) = 0\)
- At \((1,0)\): \(F = \beta \hat{y}\)
- At \((2,0)\): \(F = 2\beta \hat{y}\)
- At \((0,1)\): \(F = \alpha x \hat{x} + \beta x \hat{y}\)
- At \((1,1)\): \(F = \alpha x \hat{x} + 2\beta \hat{y}\)

Now let's take point A to be the origin and point B (destination) as \((1,1)\). Compute the work done by \(F\) on an object moving from A to B on the paths shown on the figure.

**Path 1:** Any path can be divided into many infinitesimal displacement vectors. In this problem, path 1 is a straight line connecting \((0,0)\) to \((1,1)\). As a very crude approximation we can imagine going directly from A to B in one step and take the work of the force to the value of the beginning of the path.

\[ \Delta r = 1\hat{i} + 1\hat{j} \quad F(0,0) = 0 \hat{r} + 0 \hat{j} \]

So \(W = 0\) (First approximation)

- A better approximation would be if we divided the path into two equal pieces.

\[ W = W_1 + W_2 = \overrightarrow{F}(0,0) \cdot \Delta \hat{r}_1 + \overrightarrow{F}(1/2) \cdot \Delta \hat{r}_2 \]

\[ \overrightarrow{F}(0,0) = 0 + \left( \frac{\alpha \hat{i} + \beta \hat{j}}{2} \right) \cdot \left( \frac{1}{2} \hat{i} \right) \]

\[ \Delta r_1 = \frac{1}{2} \hat{i} + \frac{1}{2} \hat{j} \]

\[ \overrightarrow{F}(1/2) = \frac{\alpha \hat{i}}{2} + \frac{\beta \hat{j}}{2} \]

\[ \Delta r_2 = \frac{1}{2} \hat{i} + \frac{1}{2} \hat{j} \]

- An even better approximation would be if we divided the path into three segments:

\[ \Delta r_1 = \Delta r_2 = \Delta r_3 = \frac{1}{3} \hat{i} + \frac{1}{3} \hat{j} \]

(Important note: For a curved path all these \(\Delta r_i\)'s could be different from each other.)

\[ W = \overrightarrow{F}(0,0) \cdot \Delta r_1 + \overrightarrow{F}(1/3) \cdot \Delta r_2 + \overrightarrow{F}(2/3) \cdot \Delta r_3 \]

\[ = 0 + \left( \frac{\alpha \hat{i} + \beta \hat{j}}{3} \right) \cdot \left( \frac{1}{3} \hat{i} + \frac{1}{3} \hat{j} \right) + \left( \frac{\alpha \hat{i} + \beta \hat{j}}{3} \right) \cdot \left( \frac{2}{3} \hat{i} + \frac{1}{3} \hat{j} \right) \]

\[ = \overrightarrow{F}(\alpha \hat{i} + \beta \hat{j}) \cdot \frac{1}{3} \hat{i} + \frac{1}{3} \hat{j} \]

\[ \approx (\alpha + \beta) \cdot \frac{1}{3} \hat{i} + (\alpha + \beta) \cdot \frac{1}{3} \hat{j} \]

(Third approximation)

- The exact value can be obtained by dividing the path to infinitely many segments. But now we need a method to label these tiny segments. Here we will try two different labellings & show that we get the same results no matter which.

**Labelling 1:** Let us define a variable \(p\) which measures the distance between the origin & the starting point of the segment. The next segment will have the label \(p + dp\).

\[ p = \overrightarrow{r} \cdot \overrightarrow{dr} \]

As \(p\) measures the distance to the origin, the tiny displacement between the segments \(p\) and \(p + dp\) should have the magnitude \(dp\), i.e., \(1\Delta r = dp\)

\[ \overrightarrow{F}(\text{segment}^0) = \overrightarrow{F}(p, \cos \phi, p, \sin \phi) \]

\[ \overrightarrow{F} \cdot \overrightarrow{dr} = (p, \sin \phi, \hat{r} + \beta \cos \phi, \hat{j}) \cdot (dp, \cos \phi \hat{r} + dp, \sin \phi \hat{j}) \]

\[ dW = p \sin \phi \cos \phi \cdot \overrightarrow{dr} \cdot dp \]

\[ W = \int_0^\infty p \sin \phi \cos \phi \cdot \frac{dp}{2} = \frac{\alpha \hat{i} + \beta \hat{j}}{2} \cdot \frac{dp}{2} \]

\[ = \frac{\alpha \hat{i} + \beta \hat{j}}{2} \]

\[ = \frac{1}{\sqrt{2}} \]
Work done by force fields - continued:
- Work on path 1 (straight from (0,0) to (1,1)) is computed again, but with a different labelling.

**Labelling 2:** Let us label each infinitesimal displacement with the x-coordinate of its starting point. Since the next tiny displacement will then start at x + dx, the x-component of \( \frac{dr}{dx} \) should be dx \( \hat{i} \).

Using trigonometry: \( \vec{dr} = dx \hat{i} + tan \phi \cdot dx \hat{j} \)

We note that \( tan \phi \cdot dx \) is actually equal to dy at that particular point. So even if the path was not straight, we could use this labelling and write \( \vec{dr} = dx \hat{i} + dy \hat{j} \) where \( dy = f'(x) \cdot dx = dy = \frac{dx}{dx} \cdot dx \).

\[
\vec{F} \cdot \vec{dr} = \vec{F}(x, \tan \phi \cdot x) \cdot (dx \hat{i} + tan \phi \cdot dx \hat{j})
\]

\[
= (x \cdot \tan \phi \cdot \hat{x} + \beta \cdot x \cdot \hat{j}) \cdot (dx \hat{i} + tan \phi \cdot dx \hat{j})
\]

\[
dW = (x + \beta) \cdot tan \phi \cdot x \cdot dx
\]

W = \( \int_{0}^{1} (x + \beta) \cdot tan \phi \cdot x \cdot dx = \frac{x + \beta}{2} \).

So: We have shown in two different ways that the work done by \( \vec{F}(x,y) \) over path 1 is equal to \( \frac{x + \beta}{2} \).

**Path 2:** This path consists of two straight line segments: from (0,0) to (1,0) and then from (1,0) to (1,1).

\[
W = \int_{0}^{1} F \cdot dr + \int_{1}^{1} F \cdot dr
\]

\[
= \int_{0}^{1} (\beta \cdot \hat{j}) \cdot (dx \hat{i}) + \int_{1}^{1} (xy \hat{i} + \beta \cdot \hat{j}) \cdot (dy \hat{j})
\]

No surprise as the force is always perpendicular to displacement along the x-axis.

\[
W = \beta \Rightarrow \text{A different work done over path 2 as compared to path 1.}
\]

**Forces like the one in the example are called non-conservative forces.** Depending on which path we take to reach a destination, non-conservative forces do different amounts of work.

Equivalently we can say that if a force does non-zero work over a closed path, it is non-conservative. For example, in the previous problem if we decided to go along path 2 first and then from (1,1) went back to the origin in the opposite direction of path 1, we would see that \( \vec{F} \) does non-zero work:

\[
W_{\text{closed loop}} = W_{(0,0) \to (0,1)} + W_{(1,0) \to (1,1)} + W_{(1,1) \to (0,0)}
\]

So for values of \( \alpha + \beta \), the force in the previous problem is clearly non-conservative.

**Trick question:** If \( \alpha = \beta \), can we say that \( \vec{F} = x \hat{i} + \beta \cdot x \hat{j} \) is conservative, just based on the calculations we have done so far?

Answer: No! We need to make sure that not just the two paths we studied, but all possible paths yield the same answers.

All closed loops we can draw in the xy plane must yield zero work!

**Extra bonus topic:** How can we check all possible paths? There are infinitely many of them to check!

Answer: Every closed path can be represented as the sum of infinitely many infinitesimal loops.

So if we could show that for all \( (x,y) \) the infinitesimal loop around that point yields zero work, we are done.

In general \( \vec{F}(x,y) = f(x,y) \hat{i} + g(x,y) \hat{j} \):

\[
W = (f(x,y) \hat{i} + g(x,y) \hat{j}) \cdot (dx \hat{i} + dy \hat{j})
\]

\[
= (f(x,y) + g(x,y)) \cdot dx + (f(x,y) + g(x,y)) \cdot dy
\]

\[
= \int f(x,y) \cdot dx + \int g(x,y) \cdot dy
\]

To make \( W = 0 \) \( \Rightarrow \) \( f(x,y) \cdot dx + g(x,y) \cdot dy = 0 \)

\[
\frac{dy}{dx} = -\frac{f(x,y)}{g(x,y)} \Rightarrow \text{Condition for conservative force field.}
\]
Ex/ is $F(x,y)= -y \hat{j} + \beta x \hat{s}$ conservative?

$$f(x,y) = xy \quad \frac{\partial f}{\partial y} = x$$

$$g(x,y) = \beta x \quad \frac{\partial g}{\partial x} = \beta$$

If $x = \beta$, $F$ is conservative. If $x \neq \beta$, it is not conservative.

**Question:** Is friction conservative? The answer is no. Consider we draw a circle on the board with our boardmarker. If there were no friction, the tip of the marker would be sliding smoothly on the surface & no ink would be deposited. The fact that we could displace so much ink, is an indicator that friction did non-zero work over the closed path.

In general for most cases of friction, particularly for kinetic friction, the displacement is in the opposite direction of the frictional force so a net positive work is done even on closed paths.

**Question:** Is gravity conservative? Clearly yes, it does not depend on position at all. $F(x,y) = -mg \hat{j}$ Any displacement along the horizontal direction will yield zero work as it will be perpendicular to $-mg \hat{j}$.

Remember that irrespective of the angle of the inclined plane, the speed of the object at the bottom was the same (when we have no friction).

**Question:** Is the spring force conservative? The spring force does depend on position ($-kx$), but being a 1D force field it is conservative.

[Challenge question: Show that 1D force fields are all conservative.]

**Potential Energy**

Let us imagine we choose a point arbitrarily as an origin, as a reference point. The nice feature of all conservative force fields is that for any point $A$ we choose, the work done by the field to carry an object from point $A$ to the origin will be a given value, independent of the path taken. Alternatively, if we take an object from the reference point and try to move it to an arbitrary point $A$, the work we need to do, i.e. the energy we need to impart to the system to overcome the force field is independent of path.

This allows us to assign a scalar value to each arbitrary point in space, the value of that external work done to overcome the force field. That value is known as the potential energy.

$$U_A = \int_{x}^{x} F_{ext} \cdot ds = -\int_{x}^{x} F \cdot dr$$

Ex/ Let us take the reference as the relaxed position/length of the spring.

$$F_{spring} = -kx$$

$$W_{ext} = \int_{-x}^{x} (-kx) dx = \frac{1}{2} k x^2 = U(x)$$

Thanks to the conservative nature of the spring force, it is very easy to compute how much work needs to be done to get the spring from $x_1$ to $x_2$:

$$\Delta U = U(x_2) - U(x_1) = \frac{1}{2} k (x_2^2 - x_1^2)$$

Or imagine a spring mass system on a frictionless surface. If the speed of the mass is $v$ at $x=0$, how far can it reach max?

$$\frac{1}{2} k x_{max}^2 = \frac{1}{2} m v^2 \implies x_{max} = \sqrt{\frac{m}{k}} v$$

Ex/ What is the work that we need to do to lift an object of mass $m$ to a height $h$ above the ground?

$$W_{ext} = (mg) \cdot (h \hat{j}) = mgh$$

What is the work done by gravity during this process? $W_{g} = (mg) \cdot (h \hat{j}) = -mgh$

What is the total work done on the object?

(Assuming no change in its speed & no friction)

$$W_{total} = W_{ext} + W_{g} = 0$$

So the energy of the object did not change!

Where did it go?

Answer: Into the invisible spring that connects the object to the Earth, like the spring in the previous example. The energy is stored in the gravitational field!